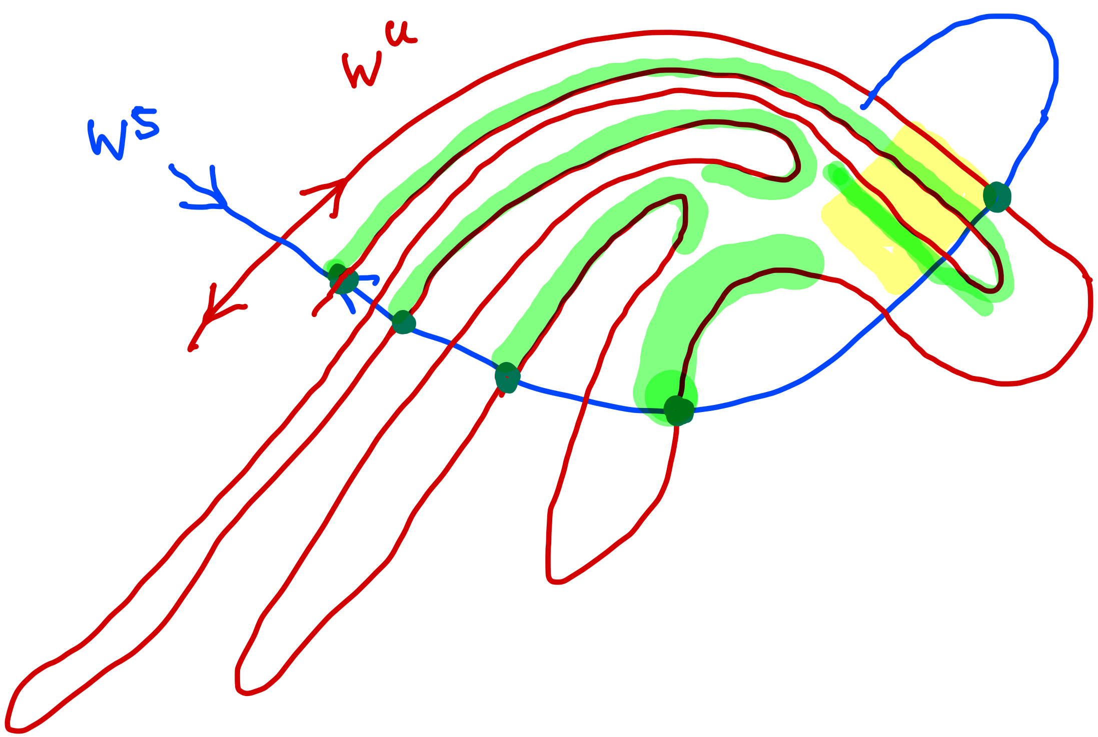
4.4. Smale's horseshoe map and symbolic dynamics



region R

$\downarrow$ 4 iterations

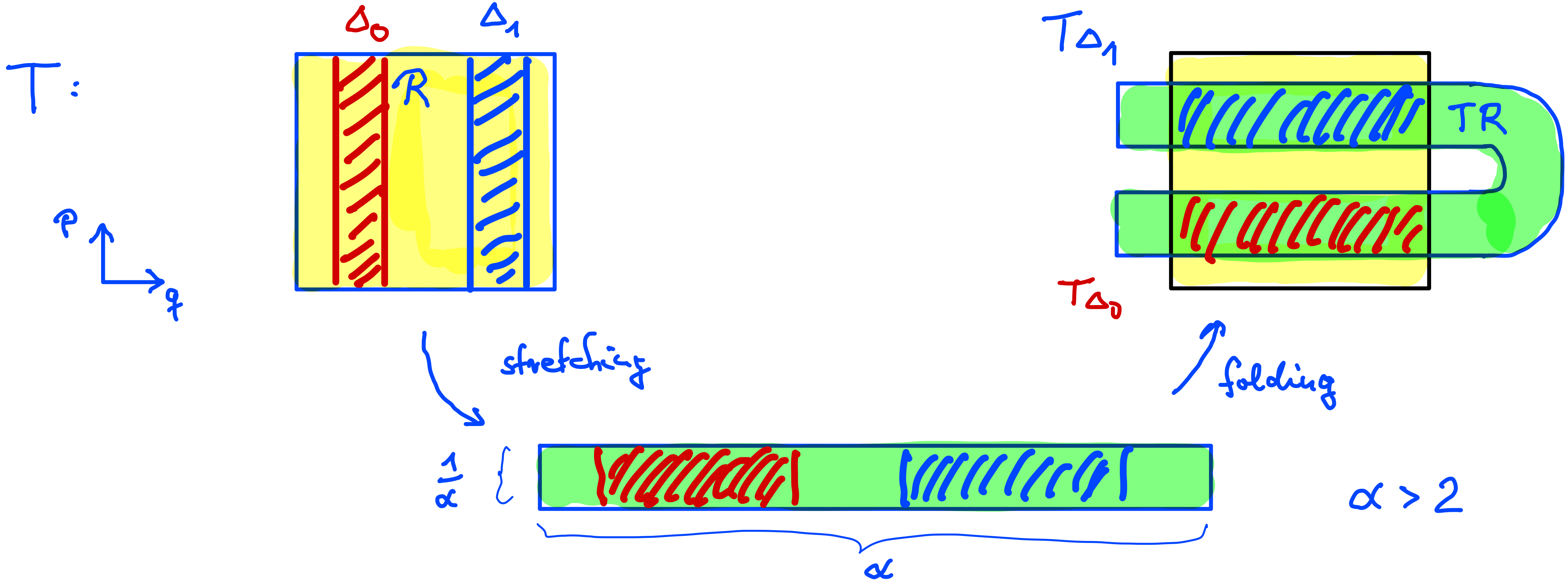
horseshoe intersecting with R

aim: simple model for stretching and folding

$\rightarrow$ understand complex dynamics

Def.: Smale's horseshoe map T

T :



Consider dynamics of points remaining in R:

$$\begin{pmatrix} q \\ p \end{pmatrix} \in \Delta_0 : \quad T\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \alpha q + \gamma_1 \\ \frac{1}{\alpha} p + \gamma_2 \end{pmatrix}$$

$$\begin{pmatrix} q \\ p \end{pmatrix} \in \Delta_1 : \quad T\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -\alpha q + \gamma_3 \\ -\frac{1}{\alpha} p + \gamma_4 \end{pmatrix}$$

Def.: trapped set $\Lambda := \left\{ x = \begin{pmatrix} q \\ p \end{pmatrix} \in R : T^n x \in R \quad \forall n \in \mathbb{Z} \right\}$

T^{-1} for $T\Delta_0$, $T\Delta_1$ defined similarly

aims: i) properties of Λ, ii) properties of orbits in Λ, iii) generalization

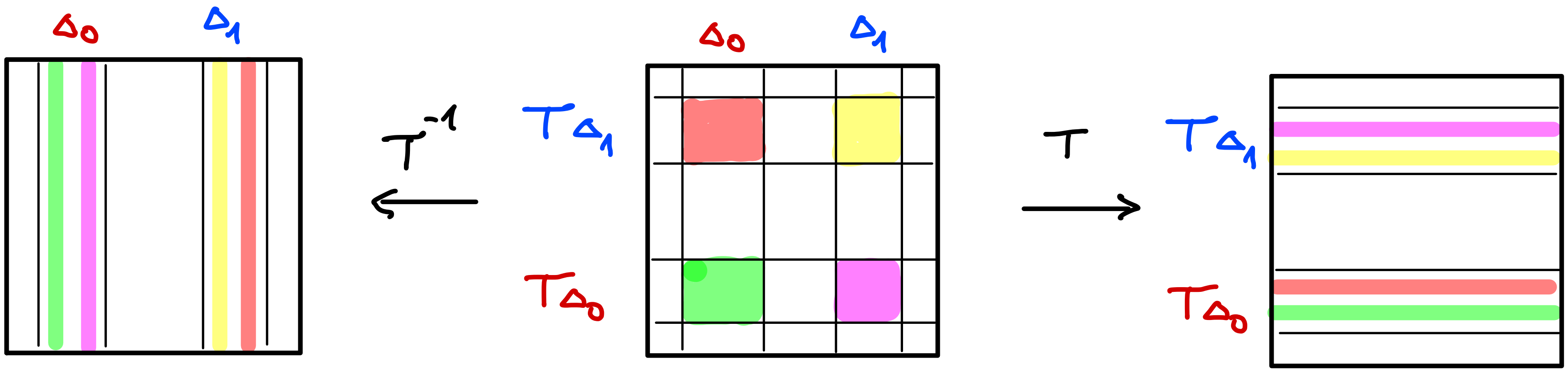
i) Properties of trapped set Λ

$$\Lambda := \left\{ x = \binom{q}{p} \in R : T^n x \in R \quad \forall n \in \mathbb{Z} \right\}$$

- one step approximation (forward and backward)

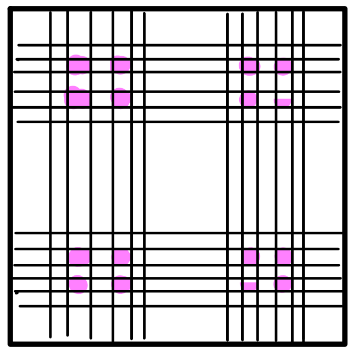
$$\Lambda_1 := \left\{ x = \binom{q}{p} \in R : Tx \in R \text{ and } T^{-1}x \in R \right\} = \bigcap_{n=-1}^{1} T^n(R)$$

$T^{-1}(\Lambda_1)$ $\qquad\qquad$ Λ_1 $\qquad\qquad$ $T(\Lambda_1)$



- two step approximation (forward and backward)

$$\Lambda_2 := \left\{ x = \binom{q}{p} \in R : T^n x \in R \; ; \; -2 \leq n \leq 2 \right\} = \bigcap_{n=-2}^{2} T^n(R)$$



- general structure of trapped set $\Lambda = \bigcap_{n \in \mathbb{Z}} T^n(R)$

- invariance: $T^{\pm 1}(\Lambda) = \bigcap_{n \in \mathbb{Z}} T^{n \pm 1}(R) = \bigcap_{n \in \mathbb{Z}} T^n(R) = \Lambda$

- Λ is hyperbolic : all points of Λ are hyperbolic, i.e. expanding and contracting direction

$\Rightarrow$ trapped set Λ : hyperbolic invariant set (chaotic saddle)

- Λ is of measure zero : $\mu(\Lambda) = 4^n \cdot \left(\frac{1}{\alpha^2}\right)^n = \left(\frac{2}{\alpha}\right)^{2n} \xrightarrow[\alpha > 2]{n \to \infty} 0$

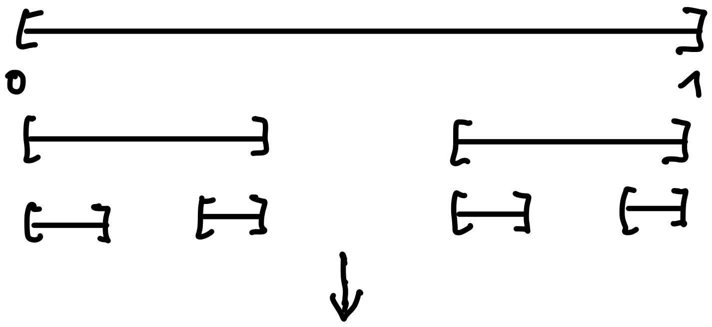
- Λ is a Cantor set

Def: Cantor set:

- closed (all accumulation points within set)
- nowhere dense (1D: no intervals, everywhere gaps)
- nowhere isolated (every point has in its neighborhood other points)

example: Middle Third Cantor set

construction:



1

$\frac{1}{3} \cdot 2$

$\frac{1}{3^2} \cdot 4$

Middle Third Cantor set $\quad \left(\frac{1}{3}\right)^n \cdot 2^n = \left(\frac{2}{3}\right)^n \xrightarrow[n \to \infty]{} 0$

zero measure, uncountable

fractal dimension : How many boxes N of size ε needed to cover?

$$N(\varepsilon) \sim \varepsilon^{-D}$$

in typical cases

D: box counting dimension (= Hausdorff dimension)

Middle Third Cantor set : $\varepsilon = \left(\frac{1}{3}\right)^n$; $N(\varepsilon) = 2^n = \varepsilon^{-D} \Rightarrow D = \frac{\ln 2}{\ln 3} = 0.6309$

Λ of horseshoe map : $\varepsilon = \left(\frac{1}{\alpha}\right)^n$; $N(\varepsilon) = 4^n = \varepsilon^{-D} \Rightarrow D = \frac{\ln 4}{\ln \alpha} = 2\frac{\ln 2}{\ln \alpha} < 2$

ii) Symbolic dynamics on Λ

dynamics on Λ:

initial condition: $x_0 \in \Lambda$

orbit: $(x_n)_{n \in \mathbb{Z}}$ with $x_n := T^n x_0$

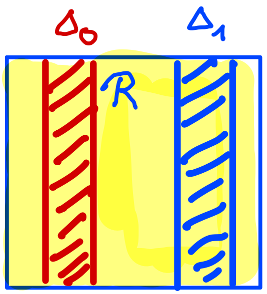
$\ldots, x_{-1}, x_0, x_1, x_2, \ldots$

$\updownarrow$

symbolic orbit: $\ldots, i_{-1}, i_0, i_1, i_2, \ldots$ $\quad i$

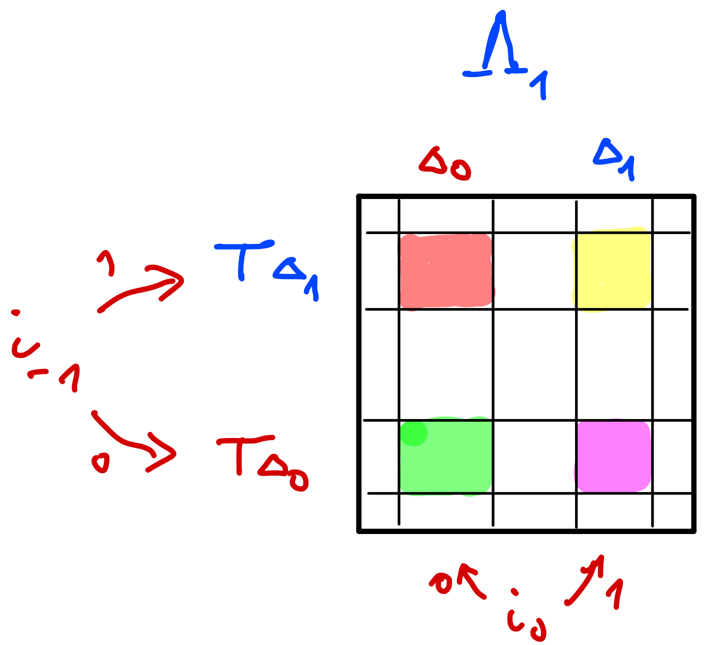
binary number $\quad 0 \; 0 \; 1 \, . \, 0 \; 1 \; 1 \; 0$

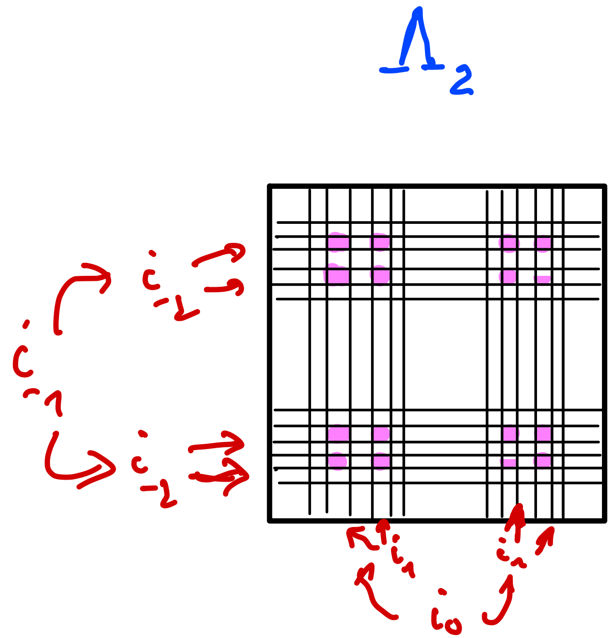
symbol $i_n := \begin{cases} 0 \,; & x_n \in \Delta_0 \\ 1 \,; & x_n \in \Delta_1 \end{cases}$ $\quad$ unique, as $\Delta_0 \cap \Delta_1 = \emptyset$



$x_0 \longrightarrow i$ unique

$i \longrightarrow x_0$ unique?





$$M_{i_{-1}.i_0} = \{x \in R \mid T^k x \in \Delta_{i_k} \, , -1 \le k \le 0\}$$

4 squares

2^2 symbolic orbit of length 2

$$M_{i_{-n}\dots i_{n-1}} = \{x \in R \mid T^k x \in \Delta_{i_k} \, , -n \le k \le n-1\}$$

4^n squares

2^{2n} symbolic orbit of length $2n$

$n \to \infty$: • square $M_{i_{-n}\dots i_{n-1}}$ converges to some point x_0 with $T^k x_0 \in \Delta_{i_k}$ $\forall k$

• no other point $y_0 \in R$ has this property

$\Rightarrow$ $i \longrightarrow x_0$ unique!

$\Rightarrow$ one-to-one correspondence of

elements in Λ and symbolic orbit

implications:

1. mapping continuous: nearby points in Λ have symbolic orbits which are identical over large regions

2. dynamics: map T on Λ $\longleftrightarrow$ shift of symbols

Def. shift S: $S\,i = j$ with $j_n = i_{n+1}$

i) $\exists$ periodic orbits with arbitrary periods

e.g.: $\ldots 011\ 011.011\ 011 \ldots$ $\qquad T^3 x = x$

ii) periodic orbits are dense within set of all orbits

given $x_0 \in \Lambda$ and its symbolic sequence i

construct $j = \begin{cases} j_n = i_n \text{ for } -m \leq n \leq m-1 \\ \text{other } j_k \text{ by periodic extension} \end{cases} \rightarrow$ period sym. orbit j

$\rightarrow$ periodic orbit

iii) $\exists$ orbits dense in Ω

symbolic orbit from all possible finite symbol sequences:

$\ldots\; 0\; 1\; 00\; 01\; 10\; 11\; 000\; 001\; 010\; 011\; \ldots$

iv) practical indeterminism

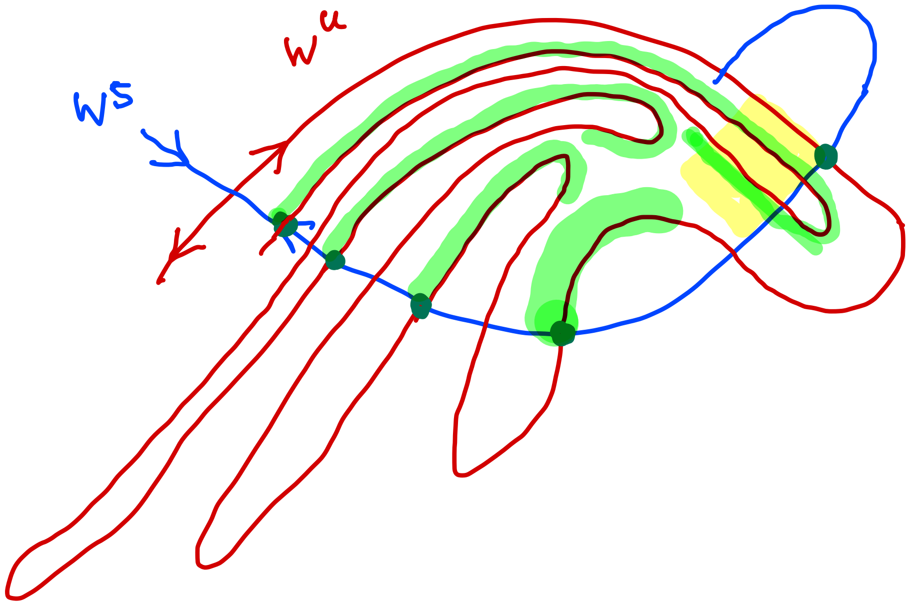
Let x_0 be known with precision α^{-n}, i.e. $x \in M_{i_{-n} \ldots i_{n-1}}$

$\Rightarrow$ $i_{-n}, \ldots, i_{n-1}$ known, but not $i_n, i_{n+1}, \ldots$

$\Rightarrow$ prediction error $O(1)$ for x_n, x_{n+1}

v) dynamics corresponds to coin flipping

iii) generalization



Smale-Birkhoff homoclinic theorem

Let T be a diffeomorphism with a hyperbolic fixed point and a transverse homoclinic point

$\Rightarrow$ 1. $\exists$ a hyperbolic invariant set Λ

2. T on Λ is equivalent to a shift on symbolic orbit

3. Λ is a Cantor set

$\Rightarrow$ · infinite number of periodic orbits

· sensitive dependence