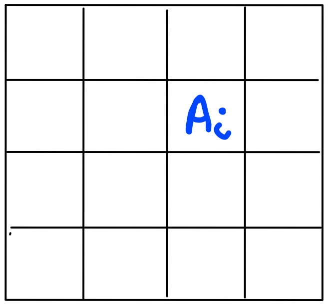
4.5.3. Kolmogorov - Sinai entropy

How to quantify sensitive dependence on initial conditions ?

idea: repeated approximate measurements
give more new information
for „chaotic" dynamics than for „regular" dynamics

rigorous approach: partitions and how they change under dynamics

Def.: partition α :



N regions with measure $\mu_i = \mu(A_i)$

$\sum_i \mu_i = 1$

Def.: entropy of partition α :

$$h(\alpha) = - \sum_{i=1}^{N} \mu_i \ln \mu_i$$

Def.: entropy of partition α relative to map T:

$$h(\alpha, T) = \lim_{n\to\infty} \frac{1}{n} h(\alpha \vee T\alpha \vee T^2\alpha \vee \dots \vee T^{n-1}\alpha)$$

$\alpha \vee T\alpha$: combined partition, up to N^2 regions

$\alpha \vee \dots \vee T^{n-1}\alpha$: up to N^n regions

Def.: Kolmogorov-Sinai entropy (or metric entropy) of map T

$$h(T) = \sup_{\alpha} h(\alpha, T)$$

remarks:
- $h(T) \geq 0$
- $h(T) > 0$ is rigorous version of sensitive dependence on initial cond.
- numerically inconvenient
- linear map on torus (cat map): $h = \sum_{\lambda_i > 0} \lambda_i$ (Lyapunov exponents, 4.5.4)
- Baker map $\leftarrow$ exercise

Def.: K-system : $h(T) > 0$

- very few examples : Sinai billiard
  Bunimovich stadium billiard

Def.: C-system (Anosov) : hyperbolic at <u>every</u> point

- example: cat map
- not Sinai billiard nor Bunimovich stadium billiard:
  family of bouncing ball orbits (measure zero) is not hyperbolic

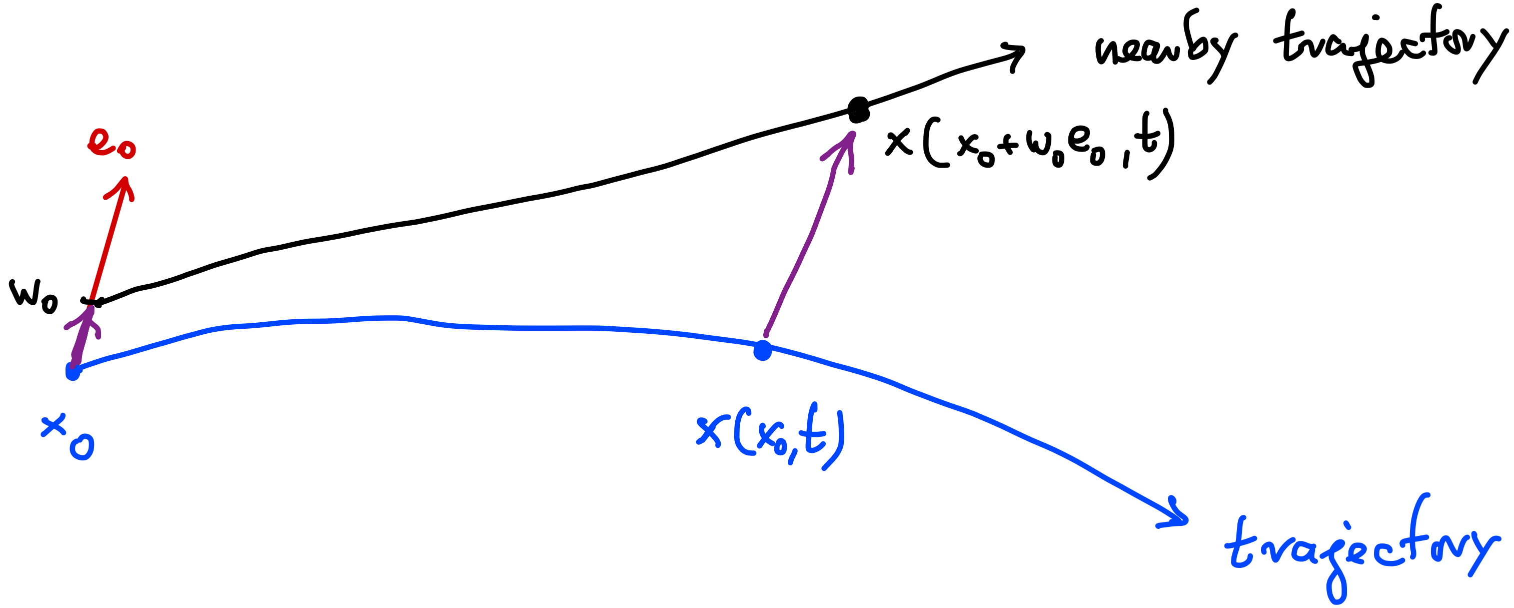
Def.: Bernoulli system : equivalent to coin flip

- example: Baker map

one can show: Bernoulli $\Rightarrow$ C-system $\Rightarrow$ K-system $\Rightarrow$ mixing $\Rightarrow$ ergodic

4.5.4. Lyapunov exponents

motivation:
- show exponential divergence of trajectories directly
- numerically relevant method



Def.: $$\lambda(x_0,e_0) := \lim_{\substack{w_0\to 0\\ t\to\infty}} \frac{1}{t}\ln\frac{|x(x_0+w_0e_0,t)-x(x_0,t)|}{w_0}$$

remarks:

- $\lambda>0 \;\hat{=}\;$ exponentially increasing distance $e^{\lambda t}$

- M-dim. system: M Lyapunov exponents $\lambda_i(x_0) = \lambda(x_0, e_i)$
- $\lambda_i = \lambda_i(x_0)$ are independent of x_0 for almost all x_0 within ergodic component
- ordering: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M$
- Hamiltonian system, N d.o.f.: $M = 2N$, $\lambda_i = -\lambda_{M+1-i}$

  $\Rightarrow$
  - $\sum_i \lambda_i = 0$ area conservation
  - $\lambda_N = \lambda_{N+1} = 0$:
    - along trajectory
    - perpendicular to energy shell
  - $N = 2$:
    - regular dynamics: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$
    - chaotic dynamics: $\lambda_1 > 0$ $\lambda_2 = \lambda_3 = 0$ $\lambda_4 = -\lambda_1 < 0$

- p-dim. volume element: changes exponentially with first p Lyapunov exponents

$$\lambda^{(p)} = \lambda_1 + \dots + \lambda_p$$

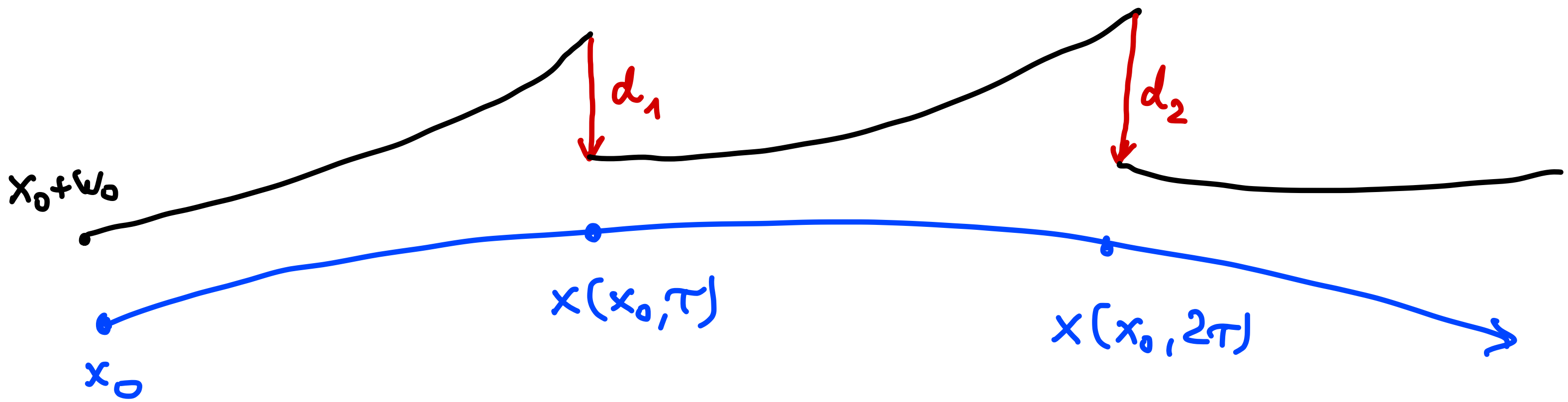
- numerical approach for λ_1

problem: $\left.\begin{array}{l} w_0 > 10^{-14} \\ |w(x_0,t)| \ll 1 \end{array}\right\}$ $\Rightarrow$ finite time t

$\Rightarrow$ λ would depend on x_0

(finite time Lyapunov exponent)

solution: reduce norm of $w(x_0,t)$ at times $\tau, 2\tau, \ldots$ to w_0 $\Rightarrow$ factor $d_j = \dfrac{|w(x_0, j\tau)|}{w_0}$



$$\lambda_1 = \lim_{k\to\infty} \frac{1}{k\tau} \sum_{j=1}^{k} \ln d_j$$

4.5.5. Shadowing Theorem

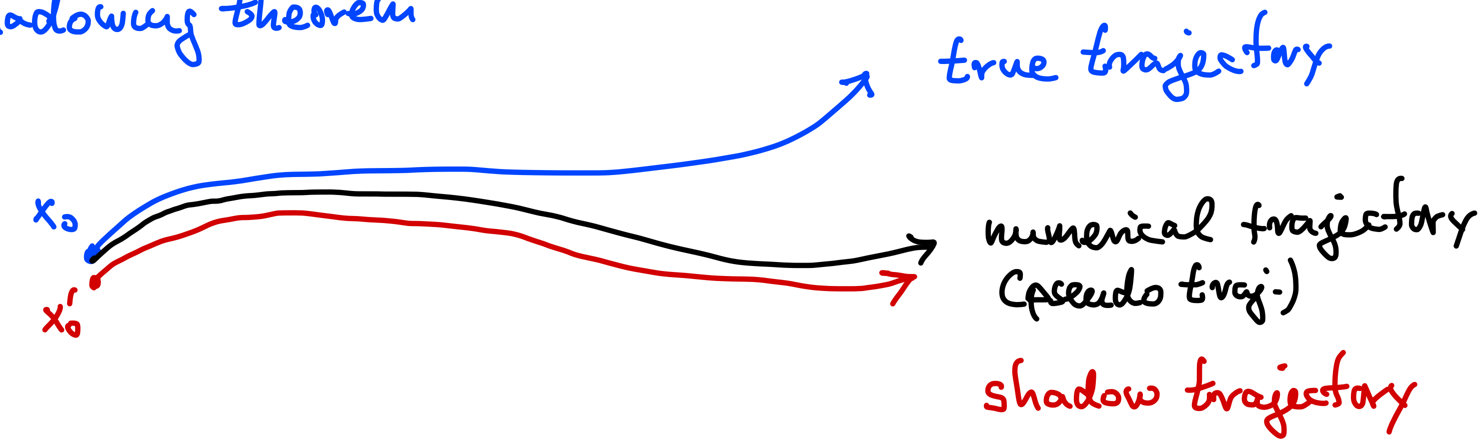
Numerics for chaotic dynamics has serious problem:
($\lambda_1 > 0$)

round-off errors lead after short time
to completely wrong trajectory

Numerics useless?

e.g. $\lambda = \ln 2$, $t = 50$: $e^{\lambda t} = 2^{50} \approx 10^{15}$

Solution: Shadowing theorem



idea: use hyperbolicity to find shadow trajectory
with slightly different initial condition x_0'
which is everywhere close to numerical trajectory

a) map contracting in all directions

∃ shadow traj.: take same initial cond. as numerical trajectory

⇒ all errors of num. traj. are contracted to shadow traj.

b) map expanding in all directions

∃ shadow traj.: take last point of numerical trajectory
iterate backwards

c) general map (e.g. Hamiltonian system)

∃ shadow traj.: use beginning of numerical trajectory
in contracting direction
and use end of numerical trajectory
in expanding direction

hyperbolic system: ∃ shadow traj. for arbitrarily long times

general system: ∃ shadow traj. at least for finite times