6. Quantum chaos – introduction

→ repeat: 1.2. What is quantum chaos?

classical	quantum
• mixed, regular, chaotic phase space	?
• homoclinic tangle	
• partial barriers	
• Lyapunov exponent	
• Kolmogorov-Sinai entropy	

6.1. Is there chaotic dynamics in quantum mechanics?

classical

chaos: exponential divergence of trajectories with arbitrarily closeby initial conditions for arbitrarily long times

quantum

wave packets

$\Delta x \, \Delta p \geq \frac{\hbar}{2}$ Heisenberg uncertainty

overlap initially:

$$\langle \psi_1(t=0) \,|\, \psi_2(t=0) \rangle$$

overlap at time t:

$$\langle \psi_1(t) \,|\, \psi_2(t) \rangle$$

$$= \langle U\psi_1(0) \,|\, U\psi_2(0) \rangle \qquad U = U(t,0)$$

$$= \langle \psi_1(0) \,|\, \underbrace{U^\dagger U}_{\mathbb{1}} \,|\, \psi_2(0) \rangle$$

$$= \langle \psi_1(0) \,|\, \psi_2(0) \rangle$$

$\Rightarrow$ q.m. overlap is time independent

(physical distance of wave packets: Wang, Wang, Wu PRE 2021)

$\Rightarrow$ no exponential divergence if initial wave packets overlap

remark: also true for classical density $\varrho(q,p,t)$ with finite width!

$\Rightarrow$ Chaos in classical mechanics needs trajectories (points in ph. sp.).

They do not exist in quantum mechanics.

But: Quantum mechanics must include classical mechanics

in the limit $\frac{h}{\underbrace{\text{typical action}}_{\text{length} \times \text{momentum}}} \longrightarrow 0$ (or short: $h \to 0$)

problem: limit is singular

Quantum wave packet follows classical trajectory for finite times

- width of wave packet increases
- Ehrenfest time τ_E: position and momentum expectation values start to deviate from classical trajectory

Ehrenfest theorem: expectation values follow equations as in classical mechanics with potential averaged over wave packet

What is maximal possible time for wave packets to follow diverging cl. traj.?

1. minimal initial distance on 2D Poincaré section: $d(0) \overset{!}{=} \sqrt{h}$
2. distance increases with Lyapunov exponent: $d(t) \approx e^{\lambda t} d(0)$
3. maximal distance due to size Γ of Poincaré section: $\sqrt{\Gamma} \overset{!}{=} d(\tau_E) = e^{\lambda \tau_E} d(0)$

$$\Rightarrow \quad \tau_E = \frac{1}{2\lambda} \ln \frac{\Gamma}{h} \qquad \left(\tau_E \to \infty \text{ for } \frac{h}{\Gamma} \to 0\right)$$

- <u>Heisenberg time τ_H:</u>

$$\tau_H = \frac{h}{\langle \Delta E \rangle}$$

← mean distance of energies

important for time evolution:

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle = \sum_n \langle \varphi_n | \psi(0)\rangle \, e^{-\frac{iE_n t}{\hbar}} |\varphi_n\rangle$$

$\underline{1\!\!1} = \sum_n |\varphi_n\rangle\langle\varphi_n|$

energy eigenstates

$H|\varphi_n\rangle = E_n|\varphi_n\rangle$

frequencies $\frac{E_n}{\hbar}$

typically irrationally related

→ quasiperiodic

wave packet probes spectrum E_n with resolution

$t < \tau_H$: individual levels not resolved

$t > \tau_H$: individual levels resolved ⇒ q.m. time evolution gives no further information

Wrong statement about quantum vs. classical: Berry 1989

Quantum mechanics shows no chaos,
because Schrödinger equation is linear,
while chaotic dynamics appears in non-linear systems.

Why wrong? Compares eqs. for wave dynamics with eqs. for trajectories

Better comparison:

q.m. probability amplitude $\psi(x,t)$

$\downarrow$ time evolution

Schrödinger eq.: $\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar} H\psi$

cl. probability density $\rho(q,p,t)$

$\downarrow$ time evolution

Liouville eq.: $\frac{\partial\rho}{\partial t} = \{\rho, H\}$

But: Heisenberg uncertainty $\longleftrightarrow$ δ-distributions in phase space possible

6.2. Time reversal

quantum dynamics

is more stable than

classical dynamics

origin: c.: rounding errors grow exponentially ($\lambda > 0$)
and destroy information about initial condition

q.m.: numerical errors grow linearly

6.3. Sensitive dependence on Hamilton operator

- fixed initial condition: $|\psi(0)\rangle$
- slightly different Hamiltonians: H_1, H_2

$$|\psi_1(t)\rangle = e^{-\frac{iH_1 t}{\hbar}} |\psi(0)\rangle$$

$$|\psi_2(t)\rangle = e^{-\frac{iH_2 t}{\hbar}} |\psi(0)\rangle$$

overlap at time t: $F(t) = |\langle \psi_1(t) | \psi_2(t) \rangle|^2$ Fidelity Peres (1984)

Loschmidt echo

cl. chaotic : $F(t)$ decays with Lyapunov exponent Pastawski, Jalabert (2001)

cl. regular : slower decay

6.4. Overview

- eigenvalue statistics, random matrix theory
- eigenfunctions
- time-periodic quantum systems: kicked rotor
  dynamical localization
- semiclassics: Gutzwiller trace formula
  orbit bunching
- quantum chaotic scattering: conductance fluctuations
  fractal Weyl law
  structure of resonance states
- experimental systems