7. Eigenvalue statistics

7.1. Introduction

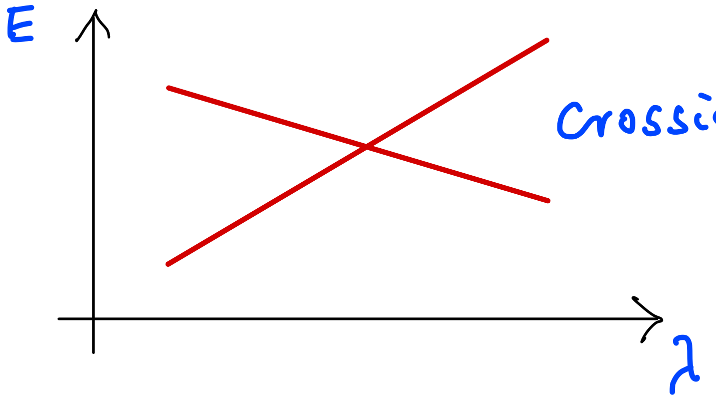
observation:

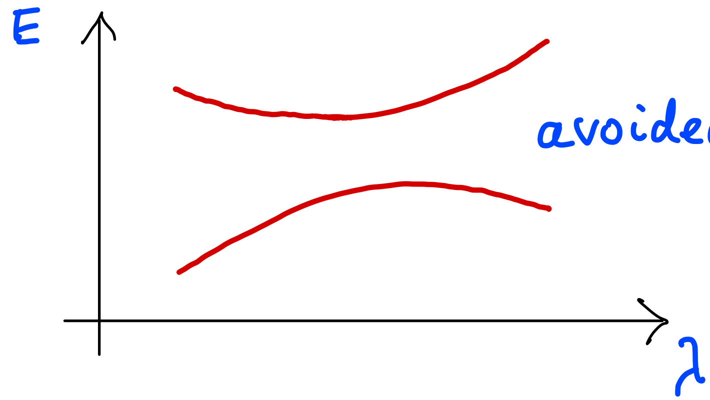
q.m. energy spectrum changes qualitatively
with change of classical dynamics

- avoided level crossings if cl. dynamics chaotic

- level crossings if cl. dynamics integrable

7.2. Level crossings





2 isolated levels: describe by hermitian 2×2 matrix

von Neumann
Wigner 1929

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = H \overset{!}{=} H^\dagger = \begin{pmatrix} H_{11}^* & H_{21}^* \\ H_{12}^* & H_{22}^* \end{pmatrix} \Rightarrow$$

H_{11}, H_{22} real

$H_{12} = H_{21}^*$ complex

4 real parameters

eigenvalues: $H\psi = E\psi \Rightarrow \det(H - E\,\mathbb{1}) = 0$

$$\Rightarrow E_{1/2} = \frac{H_{11} + H_{22}}{2} \pm \sqrt{\frac{(H_{11} - H_{22})^2}{4} + (\mathrm{Re}\, H_{12})^2 + (\mathrm{Im}\, H_{12})^2}$$

level crossing $(E_1 = E_2)$ iff $\underset{\substack{\uparrow \\ \text{codimension } 3}}{3}$ real parameters are zero: $H_{11} - H_{22}$, $\mathrm{Re}\, H_{12}$, $\mathrm{Im}\, H_{12}$

Systems with time reversal symmetry:

T time reversal operator:

- $T^2 = 1$
- $[H, T] = 0$
- antiunitary: $\langle T\varphi | T\psi \rangle = \langle \varphi | \psi \rangle^*$

(unitary $\langle U\varphi | U\psi \rangle = \langle \varphi | \underbrace{U^+ U}_{\mathbb{1}} | \psi \rangle = \langle \varphi | \psi \rangle$)

proposition: Given antiunitary operator T with $[H,T]=0$, $T^2 = 1$ one can find a basis in which H is real (without diagonalizing)

$\Rightarrow \operatorname{Im} H_{12} = 0$

$\Rightarrow$ codim. $n=2$

(proof: $|\psi_1\rangle = |\varphi_1\rangle + T|\varphi_1\rangle$
$\Rightarrow T|\psi_1\rangle = |\psi_1\rangle$
...)

Systems with time reversal symmetry: codimension $n = 2$

Systems without time reversal symmetry: codimension $n = 3$

Spin systems (4×4 matrix) : codimension $n = 5$

7.3. Nearest-neighbor level spacing distribution

Def.: $P(S)\,dS$ is probability to find a spacing in $[S, S+dS]$

normalization: $\int_0^\infty P(S)\,dS = 1$

- e.g. levels E_i : $\Delta E_i := E_{i+1} - E_i$

$$P(S) = \frac{1}{N} \sum_{i=1}^{N} \delta(S - \Delta E_i)$$

$$= \langle \delta(S - \Delta E_i) \rangle_i$$

$P(S)$ for small spacings $S \ll \langle \Delta E \rangle$:

- use result of isolated avoided crossing with some codimension n $(n=2,3,5)$

$$\Delta E = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\vec{x}^2} \quad \text{with unknown distribution } W(\vec{x})$$

$$P(S) = \int d^n x \; W(\vec{x}) \; \delta\left(S - \sqrt{\vec{x}^2}\right) \qquad \vec{y} = \frac{\vec{x}}{S}$$

$$= S^n \int d^n y \; W(S\vec{y}) \; \underbrace{\delta\left(S\left(1-\sqrt{\vec{y}^2}\right)\right)}_{\frac{1}{S}\delta\left(1-\sqrt{\vec{y}^2}\right)}$$

$$= S^{n-1} \int d^n y \; \underbrace{W(S\vec{y})}_{\text{for } S\to 0 \text{ we assume } W(0)} \; \delta\left(1-\sqrt{\vec{y}^2}\right)$$

$$\Rightarrow \boxed{P(S) \sim S^{n-1} \quad \text{for } S \to 0}$$

Berry 1981

$n \geq 2$: $P(S=0) = 0$ level repulsion remarks:

- fully chaotic systems : level repulsion

  with $\beta = n-1$: $P(S) \sim S^{\beta}$ for $S \to 0$

  3 universality classes:

  Systems with time reversal symmetry: $\beta = 1$ $P(S) \sim S$

  Systems without time reversal symmetry: $\beta = 2$ $P(S) \sim S^2$

  Spin systems (4×4 matrix) : $\beta = 4$ $P(S) \sim S^4$

- integrable systems : level crossings

  EBK quantization in 2 and more dimensions

  effectively independent levels

  $\Rightarrow$ Poisson distribution $\Rightarrow$ $P(S) \sim e^{-S}$ Berry, Tabor 1977

- mixed systems: partly crossings, partly repulsion

  $$P(S=0) = 1 - \varrho_{ch}^{2}$$ (ϱ_{ch} = chaotic phase space fraction)

$\rightarrow$ after eigenfunctions