7.4. Random matrix theory

conjecture: level statistics of classically chaotic systems
is well described by random matrix theory (RMT)
Casati, Valz-Gris, Guarneri 1980; Bohigas, Giannoni, Schmit 1983

remarks: · lots of numerical + experimental evidence

· semiclassical "proof" $\longrightarrow$ semiclassics

· intuition: no system specific information except symmetries

3 universality classes:

Systems with time reversal symmetry: $\beta = 1$

Systems without time reversal symmetry: $\beta = 2$

Spin systems (4×4 matrix) : $\beta = 4$

$\Rightarrow$ 3 groups of transformations

$O(N)$ group of real orthogonal matrices $O^T O = \mathbb{1}$

$U(N)$ " " unitary " $U^\dagger U = \mathbb{1}$

symplectic group

7.4.1. Gaussian Ensembles

2×2 matrix, $\beta = 1$: H real hermitean : $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$

aim: find prob. dist. $P(H) = P(H_{11}, H_{22}, H_{12})$

under conditions:

1. normalization : $\int_{-\infty}^{\infty} dH_{11} \int_{-\infty}^{\infty} dH_{22} \int_{-\infty}^{\infty} dH_{12}\; P(H) = 1$

2. invariance of $P(H)$ under $O(2)$: Let $H' = O^T H O$ with $O^T O = \underline{1\!\!1}$

   $\Rightarrow P(H') = P(H)$

3. independence : $P(H) = P_{11}(H_{11}) \cdot P_{22}(H_{22}) \cdot P_{12}(H_{12})$

sufficient: infinitesimal orthogonal transformation

$$O = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} \quad \text{with } \theta \ll 1 \text{ and neglect terms } O(\theta^2)$$

$$H' = O^T H O = \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$$

$$= \ldots \approx \begin{pmatrix} H_{11} + \theta(-2H_{12}) & H_{12} + \theta(H_{11} - H_{22}) \\ H_{12} + \theta(H_{11} - H_{22}) & H_{22} + \theta(2H_{12}) \end{pmatrix}$$

$$P_{11}(H'_{11}) = P_{11}(H_{11} + \theta(-2H_{12})) = P_{11}(H_{11}) + \theta(-2H_{12}) \frac{dP_{11}}{dH_{11}} + O(\theta^2)$$

$$= P_{11}(H_{11}) \left(1 + \theta(-2H_{12}) \cdot \underbrace{\frac{1}{P_{11}(H_{11})} \frac{dP_{11}}{dH_{11}}}_{\frac{d \ln P_{11}}{dH_{11}}} \right)$$

$$P_{22}(H'_{22}) = P_{22}(H_{22})\left(1 + \Theta \cdot 2H_{12} \frac{d \ln P_{22}}{d H_{22}}\right)$$

$$P_{12}(H'_{12}) = P_{12}(H_{12})\left(1 + \Theta \cdot (H_{11} - H_{22}) \frac{d \ln P_{12}}{d H_{12}}\right)$$

multiply:

$$P(H') \overset{3.}{=} P(H)\left(1 - \Theta \cdot \underbrace{\left[2H_{12}\left(\frac{d \ln P_{11}}{d H_{11}} - \frac{d \ln P_{22}}{d H_{22}}\right) - (H_{11} - H_{22}) \frac{d \ln P_{12}}{d H_{12}}\right]}_{\Rightarrow 0}\right)$$

$$\overset{2.}{=} P(H)$$

$$\Rightarrow \frac{1}{H_{12}} \frac{d \ln P_{12}}{d H_{12}} = \frac{2}{H_{11} - H_{22}}\left(\frac{d \ln P_{11}}{d H_{11}} - \frac{d \ln P_{22}}{d H_{22}}\right) = a \quad \text{some constant}$$

$$\Rightarrow \frac{d \ln P_{11}}{d H_{11}} - \frac{a}{2} H_{11} = \frac{d \ln P_{22}}{d H_{22}} - \frac{a}{2} H_{22} = b \quad \text{some constant}$$

$$\frac{d \ln P_{12}}{d H_{12}} = a H_{12} \Rightarrow \ln P_{12} = \frac{a}{2} H_{12}^2 + c \Rightarrow P_{12}(H_{12}) = c_{12}\, e^{\frac{a}{2} H_{12}^2}$$

$$\frac{d \ln P_{11}}{d H_{11}} = \frac{a}{2} H_{11} + b$$

$$\Rightarrow P_{11}(H_{11}) = c_{11}\, e^{\frac{a}{4} H_{11}^2 + b H_{11}}$$

$$\Rightarrow P_{22}(H_{22}) = c_{22}\, e^{\frac{a}{4} H_{22}^2 + b H_{22}}$$

$$\Rightarrow P(H) = C\, e^{\frac{a}{4}\left(H_{11}^2 + H_{22}^2 + 2H_{12}^2\right) + b(H_{11} + H_{22})}$$

normalization: $a < 0$

choose zero of energy: $0 \overset{!}{=} \langle E_1 + E_2 \rangle = \langle H_{11} + H_{22} \rangle = 2 \langle H_{11} \rangle \Rightarrow b = 0$

$$\boxed{P(H) = \underset{\substack{\uparrow \\ \text{normalization}}}{C}\; e^{-\frac{1}{\underset{\substack{\uparrow \\ \varepsilon \text{ energy scale}}}{\varepsilon^2}} \mathrm{Tr}(H^2)}}$$

Gaussian Orthogonal Ensemble (GOE)

remarks:

- same formula for $N \times N$ matrix
- " " " unitary transformation (GUE)
- " " " symplectic " (GSE)
- explicitly for GOE: $\mathrm{Tr}(H^2) = \sum_{i=1}^{N} H_{ii}^2 + \sum_{i<j}^{N} 2 H_{ij}^2$

compare with Gaussian: $P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$ with variance σ^2

$\Rightarrow$ H_{ii} : Gaussian, mean zero, variance $\langle H_{ii}^2 \rangle = \frac{\varepsilon^2}{2}$

$\Rightarrow$ H_{ij} : Gaussian, mean zero, variance $\langle H_{ij}^2 \rangle = \frac{\varepsilon^2}{4}$

numerics: K_{ij} with $\sigma^2 = \frac{\varepsilon^2}{8}$

$H_{ij} := K_{ij} + K_{ji}$

7.4.2. Eigenvalue distribution

aim: find $P(E_1, E_2)$ for GOE : real symmetric 2x2 matrix

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} = O^T \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} O \qquad \text{with} \quad O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$H_{11}, H_{22}, H_{12} \xrightarrow{\quad\quad} E_1, E_2, \theta$$

$\uparrow$

Jacobi determinant : $J = \det \dfrac{\partial(H_{11}, H_{22}, H_{12})}{\partial(E_1, E_2, \theta)} = \dots = E_1 - E_2$

$$\Rightarrow P(E_1, E_2) = C \, |E_1 - E_2| \, e^{-\frac{1}{\varepsilon^2}(E_1^2 + E_2^2)}$$

general result for $N \times N$ matrix and all symmetry classes β $(= 1, 2, 4)$

$$P(E_1, \dots, E_N) = C \cdot \left(\prod_{\substack{\mu,\nu=1 \\ \mu<\nu}}^{N} |E_\mu - E_\nu|^\beta \right) e^{-\frac{1}{\varepsilon^2} \sum_{\mu=1}^{N} E_\mu^2}$$

7.4.3. Nearest neighbor level spacing distribution $P(S)$

$$N=2: \quad P(S) = C \int_{-\infty}^{\infty} dE_1 \int_{-\infty}^{\infty} dE_2 \; \delta\big(S - |E_1 - E_2|\big) \, |E_1 - E_2|^\beta \, e^{-\frac{1}{\varepsilon^2}(E_1^2 + E_2^2)}$$

$\Delta = E_1 - E_2$

$$= C \int_{-\infty}^{\infty} d\Delta \int_{-\infty}^{\infty} dE_2 \; \delta\big(S - |\Delta|\big) \, |\Delta|^\beta \, \underbrace{e^{-\frac{1}{\varepsilon^2}\left((\Delta + E_2)^2 + E_2^2\right)}}_{\sqrt{\frac{\pi}{2}}\,\varepsilon\, e^{-\frac{\Delta^2}{2\varepsilon^2}}}$$

$$= 2C\varepsilon\sqrt{\frac{\pi}{2}} \; S^\beta \, e^{-\frac{S^2}{2\varepsilon^2}}$$

normalization: $\int_0^\infty dS \, P(S) = 1 \quad \Rightarrow \quad C(\beta)$

average spacing: $\int_0^\infty dS \, S \, P(S) = 1 \quad \Rightarrow \quad \varepsilon(\beta)$

remarks:
- agrees with $P(S) \sim S^\beta$ for $S \to 0$
- $N \times N$ matrix with $N \to \infty$ gives $\approx 1\%$ correction

Wigner surmise (2×2):

GOE $\beta=1$:

$$P(S) = \frac{\pi}{2}\, S\, e^{-\frac{\pi}{4} S^2}$$

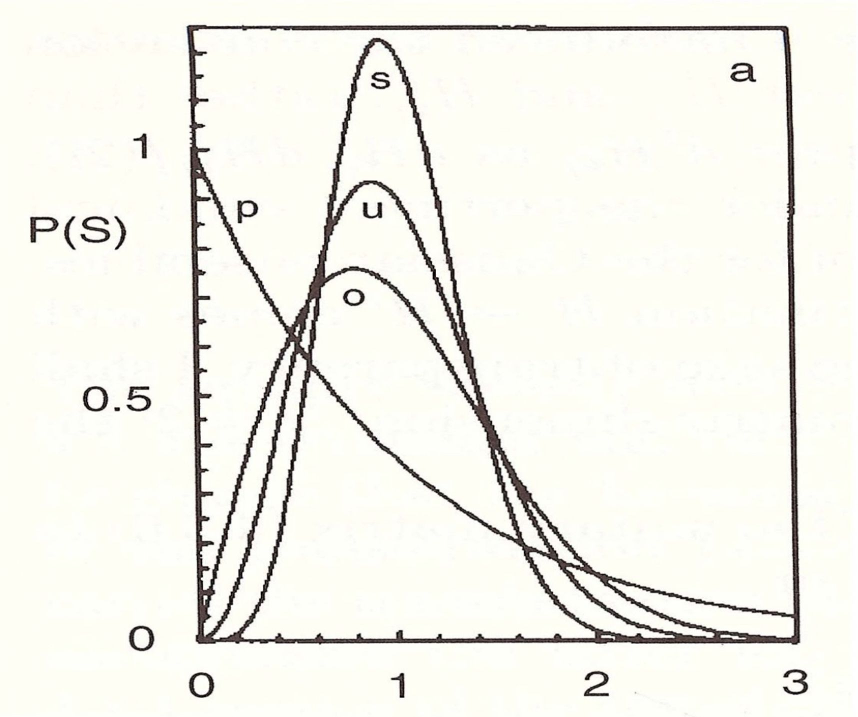
GUE $\beta=2$:

$$P(S) = \frac{32}{\pi^2}\, S^2\, e^{-\frac{4}{\pi} S^2}$$

GSE $\beta=4$:

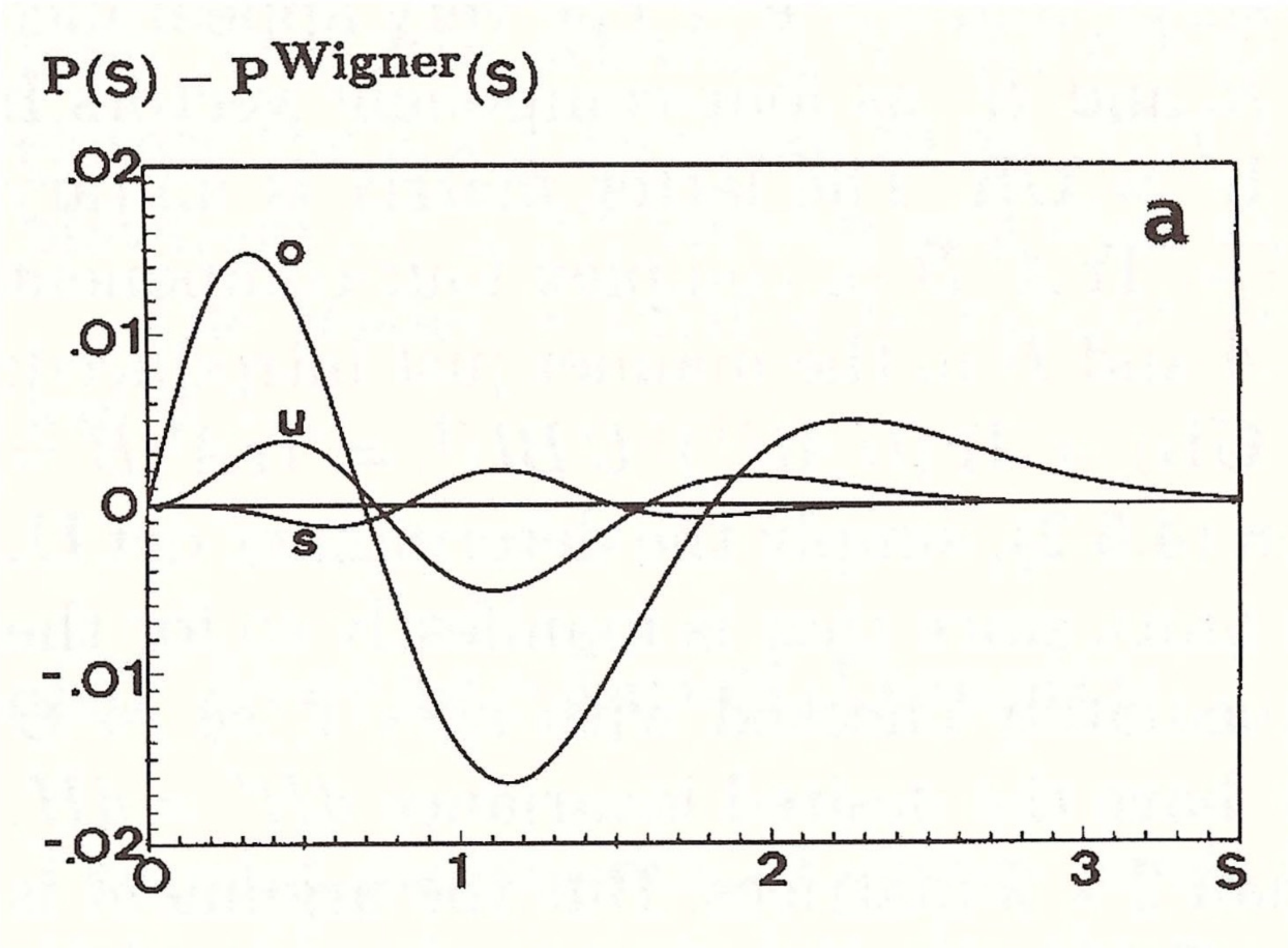
$$P(S) = \frac{2^{18}}{3^6\pi^3}\, S^4\, e^{-\frac{64}{9\pi} S^2}$$

P(S) for 2x2 matrix: GOE, GUE, GSE



Haake, Fig 4.1

Difference in P(S) for NxN matrix with N→∞ and N=2



Haake, Fig 4.2