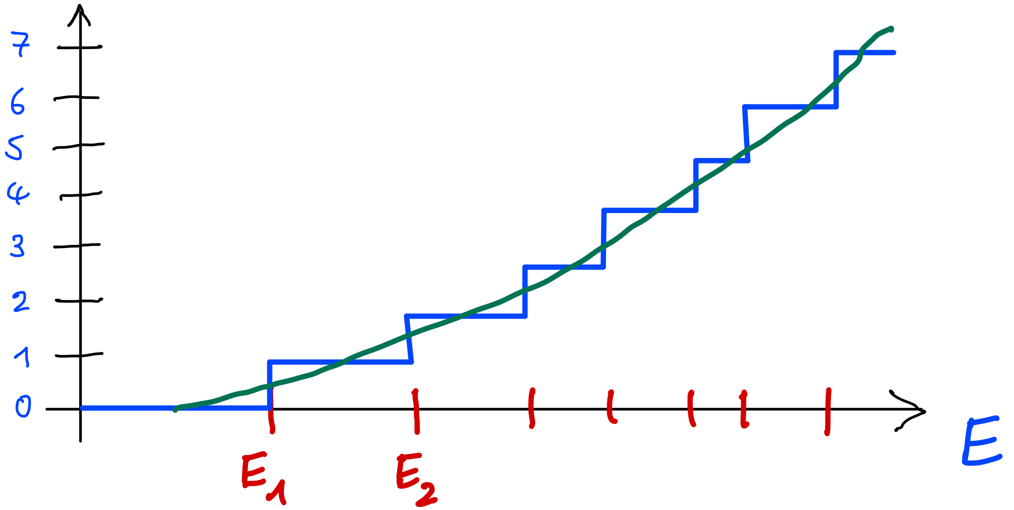
7.5. Average level density

Motivation: · Average is system specific and not related to chaotic/regular dynamics

· Universal properties appear as fluctuations on top of average

Def.: Spectral staircase (counting function): number of levels with $E_i < E$

$$N(E) = \sum_i \Theta(E - E_i)$$



Def.: Density of states: $d(E) = \frac{dN(E)}{dE} = \sum_i \delta(E - E_i)$

Def.: Average staircase $\bar{N}(E)$ smoothly approximating $N(E)$ $\Rightarrow$ $N(E) = \bar{N}(E) + N_{fl}(E)$

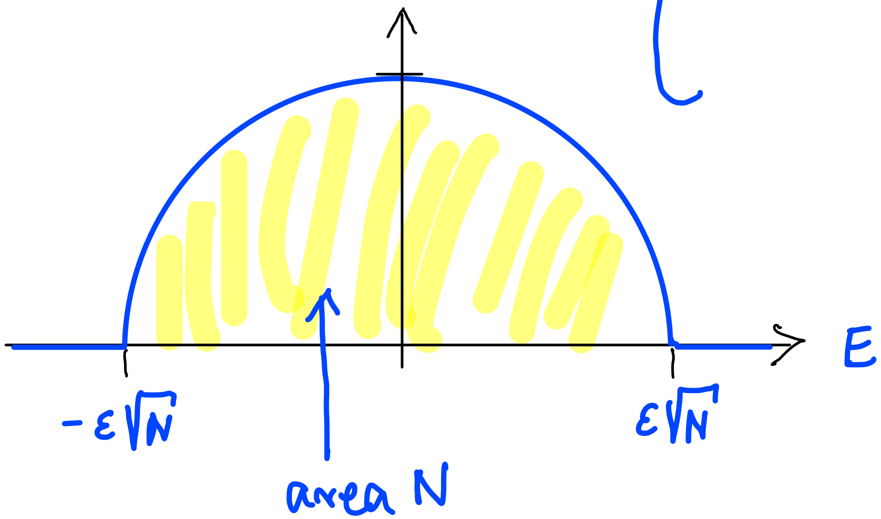
Def.: Average density of states: $\bar{d}(E) = \frac{d\bar{N}(E)}{dE}$ $\Rightarrow$ $d(E) = \bar{d}(E) + d_{fl}(E)$

What is average?

a) Random matrices: Wigner's semicircle law (1959)

$$\bar{d}(E) = N \int dE_2 \cdots dE_N \, P(E, E_2, E_3, \ldots, E_N) = \cdots =$$

$$\bar{d}(E) = \begin{cases} \dfrac{2\sqrt{N}}{\pi \varepsilon} \sqrt{1 - \left(\dfrac{E}{\varepsilon\sqrt{N}}\right)^2} & ; \quad |E| \leq \varepsilon\sqrt{N} \\ 0 & ; \quad \text{otherwise} \end{cases}$$



$$\bar{N}(E) = \int_{-\infty}^{E} dE' \, \bar{d}(E')$$

b) Hamiltonian system: Weyl law

Each quantum state occupies a Planck cell h^d

Volume of classical phase space below energy E:

$$V_\mu(E) = \int d^d p \, d^d q \; \Theta\left(E - H(\vec{q},\vec{p})\right)$$

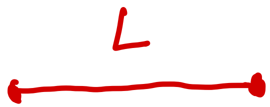
$$\bar{N}(E) = \frac{V_\mu(E)}{h^d}$$

$$\bar{d}(E) = \frac{d\bar{N}(E)}{dE} = \frac{1}{h^d} \int d^d q \, d^d p \; \delta\left(E - H(\vec{q},\vec{p})\right)$$

Example: Billiard in d dimensions: $H(\vec{q},\vec{p}) = \begin{cases} \frac{\vec{p}^2}{2m} & ; \vec{q} \text{ inside} \\ \infty & ; \vec{q} \text{ outside} \end{cases}$

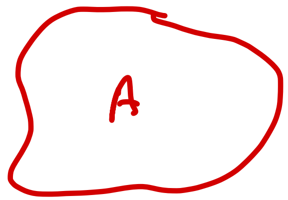
$$\Rightarrow \bar{d}(E) = \frac{V}{h^d} \int d^d p \, \delta\left(E - \frac{\vec{p}^2}{2m}\right) \underset{\vec{z} = \frac{\vec{p}}{\sqrt{2mE}}}{\overset{\uparrow}{=}} \frac{V}{h^d} \frac{(2mE)^{d/2}}{E} \underbrace{\int d^d z \, \delta(1 - \vec{z}^2)}_{\frac{1}{2} O_d \text{ surface of } d\text{-dim. unit sphere}}$$

$d=1$: $\bar{d}(E) = \frac{L\sqrt{2m/E}}{h} \sim \frac{1}{\sqrt{E}}$



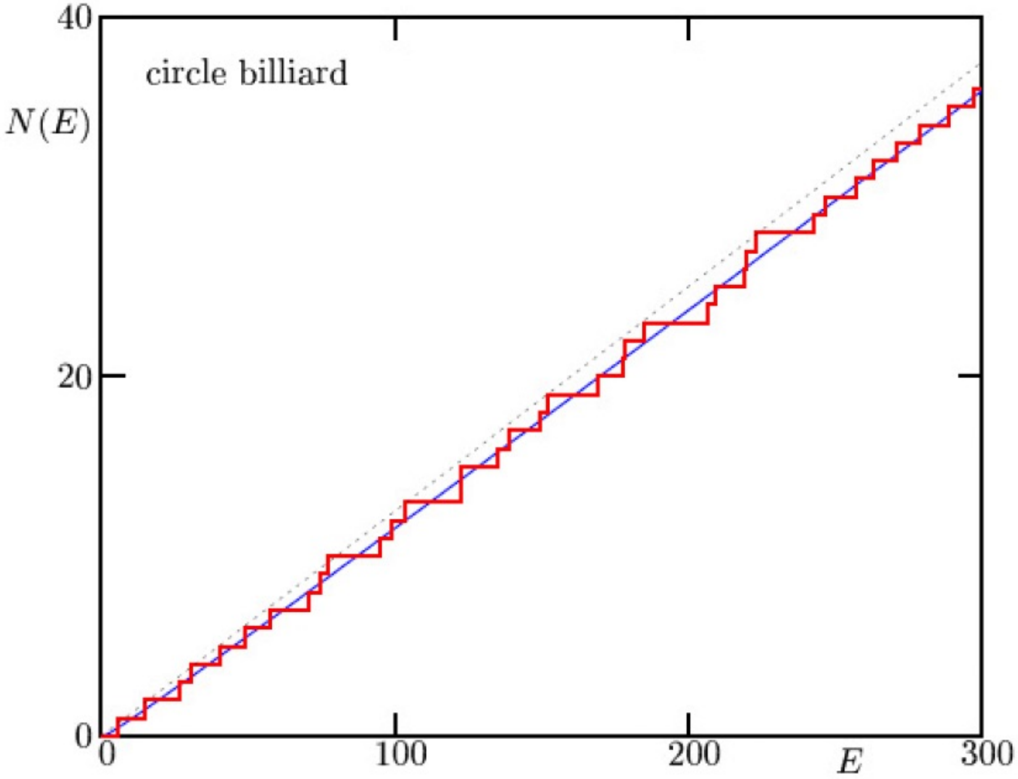
$E_n \sim n^2$

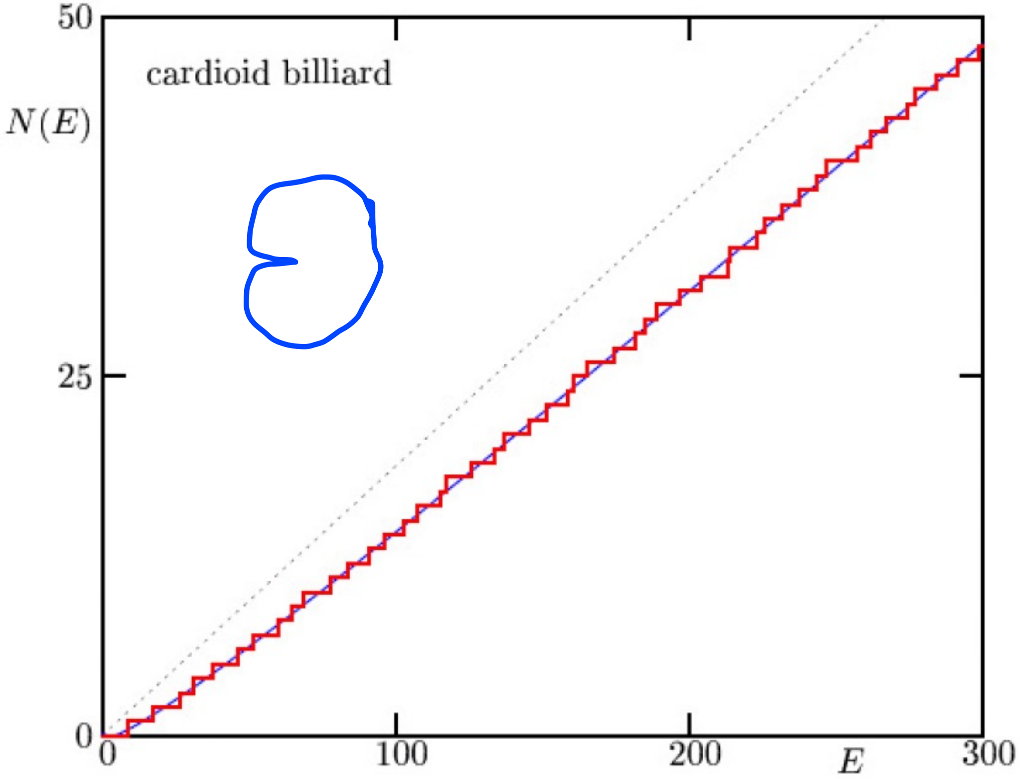
$d=2$: $\bar{d}(E) = \frac{2m}{h^2} \frac{A}{4\pi} \sim E^0$



$E_n \sim n$

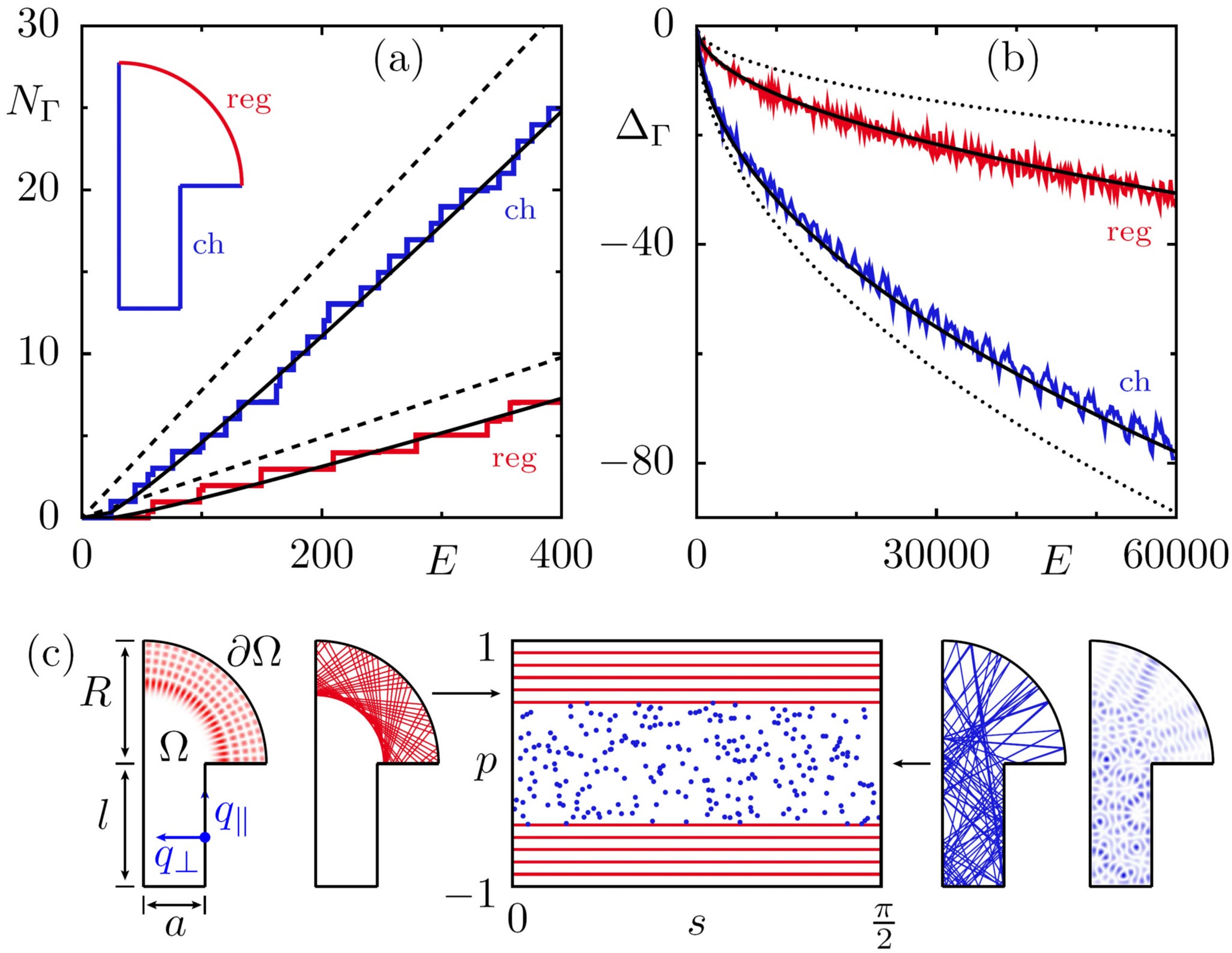
$$- \frac{2m}{h^2} \frac{L}{8\pi\sqrt{E}} + \text{corrections of higher order in } \frac{1}{\sqrt{E}}$$


circle billiard
N(E)
E
0
20
40
0
100
200
300


cardioid billiard
N(E)
E
0
25
50
0
100
200
300

A. Bäcker

Partial Weyl law:

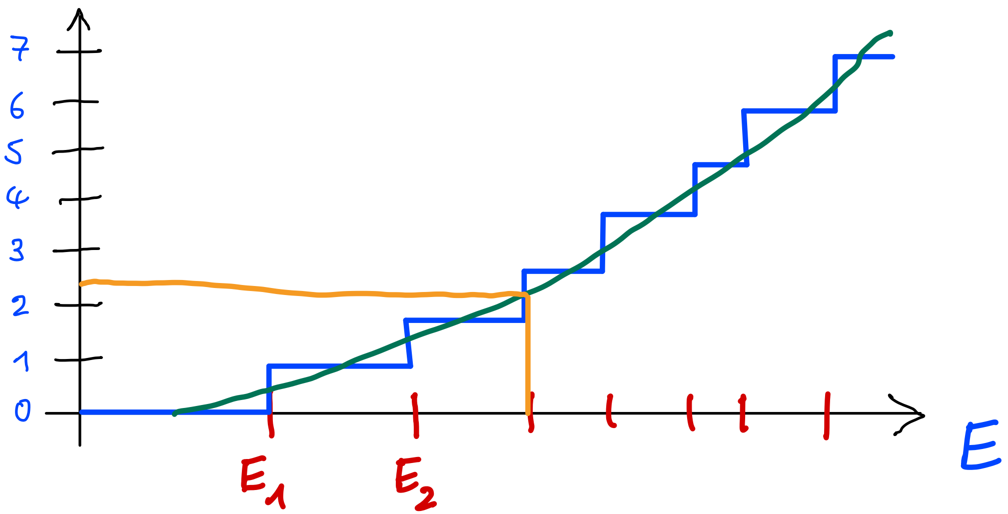


Bäcker, Ketzmerick, Löck, Schack, EPL 2011

7.6. Unfolding of spectra

Motivation: · Average is system specific and not related to chaotic/regular dynamics

· Universal properties appear as fluctuations on top of average



$$N(E) = \sum_i \Theta(E - E_i)$$

$$\bar{N}(E)$$

unfolding: $e_i = \bar{N}(E_i)$

average distance of e_i is 1 (by construction) remarks: • often $\bar{N}(E)$ or $\bar{d}(E)$ unknown $\to$ extract from data

e.g. convolution with Gaussian of width Δ

$$\bar{d}_\Delta(E) = \int_{-\infty}^{\infty} d\varepsilon \quad d(E-\varepsilon)\,\frac{e^{-\frac{\varepsilon^2}{2\Delta^2}}}{\sqrt{2\pi\Delta^2}}$$

choose Δ large, s.t. $N(E+\Delta) - N(E) \gg 1$

small, s.t. global $\bar{d}(E)$ not altered

- billiards: use Weyl law
- random matrices: use Wigner's semicircle

avoid unfolding:

- ratio of consecutive level spacings $r = \dfrac{E_{n+2} - E_{n+1}}{E_{n+1} - E_n}$

$$\tilde{r} = \min\left(r, \frac{1}{r}\right) \in [0,1]$$

$$\langle \tilde{r} \rangle_{GOE} = 4 - 2\sqrt{3} \approx 0.53\ldots \qquad \langle \tilde{r} \rangle_{Poisson} = 2\ln 2 - 1 \approx 0.39$$

7.7. Further level statistics

Motivation: $P(S)$ is not sensitive to long-range correlations

- Δ_3 statistics (rigidity): deviation of $N(E)$ from straight line

$$\Delta_3(L) = \frac{1}{L} \left\langle \min_{A,B} \int_E^{E+L} dx \left[N(x) - (Ax+B) \right]^2 \right\rangle_E = \begin{cases} \frac{L}{15} & \text{Poisson} \\ \frac{\ln L}{\pi^2} & \text{GOE} \end{cases}$$

long-range correlations

- number statistics:

$n(L)$ = number of levels in interval of length L

$$\langle n(L) \rangle = L$$

$$\langle n(L)^2 \rangle = \Sigma^2(L) = \begin{cases} L & ; \text{Poisson} \\ \frac{2}{\pi^2} \ln L & ; \text{GOE} \end{cases}$$

less fluctuations

- $E(k,L)$ = prob. to have k levels in interval of length L

- spectral two-point correlation:

  use fluctuating density of states: $d_{fl}(E) = d(E) - \bar{d}(E)$

$$Y_2(L) = \int_{-\infty}^{\infty} dE \quad d_{fl}\left(E - \frac{L}{2}\right) d_{fl}\left(E + \frac{L}{2}\right) - \delta(L)$$

prob. to have two levels with distance L (and possibly many in between)

- form factor: $K(\tau) = \int_{-\infty}^{\infty} dL \;\; Y_2(L) \;\; \cos(2\pi L \tau) + 1$