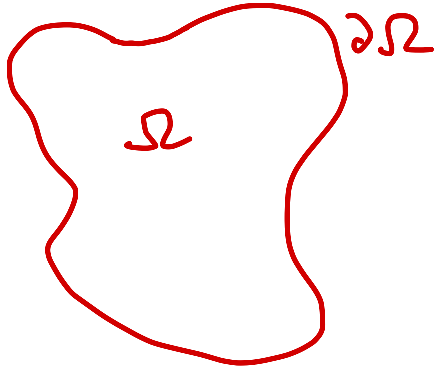
8. Eigenfunctions

8.1. Quantum billiards

- Schrödinger eq.: $\hat{H}|\psi\rangle = E|\psi\rangle$
- Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$ particle in potential
- billiard potential: $V(\vec{r}) = \begin{cases} 0 & ; \ \vec{r} \text{ inside } \Omega \\ \infty & ; \ \vec{r} \text{ outside } \Omega \end{cases}$



- position representation: $\langle \vec{r} | \psi \rangle = \psi(\vec{r})$
  - potential term: $\langle \vec{r} | V(\hat{q}) | \psi \rangle = V(\vec{r}) \, \psi(\vec{r})$
  - momentum operator: $\langle \vec{r} | \hat{p} | \psi \rangle = -i\hbar \vec{\nabla} \psi(\vec{r})$

$$\vec{r} \in \Omega : -\frac{\hbar^2}{2m} \Delta \psi(\vec{r}) = E \psi(\vec{r}) \quad \underset{E = \frac{\hbar^2 k^2}{2m}}{\Longrightarrow} \quad (\Delta + k^2)\psi(\vec{r}) = 0$$

Helmholtz eq.

e.g. in 2D: $\Delta = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$

- boundary condition: $\vec{r} \in \partial\Omega$: $\psi(\vec{r}) = 0$ (Dirichlet)

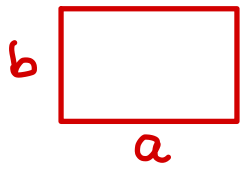
$\Rightarrow$ discrete eigenenergies $E_n > 0$ and eigenfunctions $\psi_n(\vec{r})$

8.2. Real space properties

a) integrable systems:

· rectangular billiard

$$\psi_{n_x,n_y}(x,y) = \frac{1}{N} \sin n_x \frac{\pi x}{a} \cdot \sin n_y \frac{\pi y}{b}$$

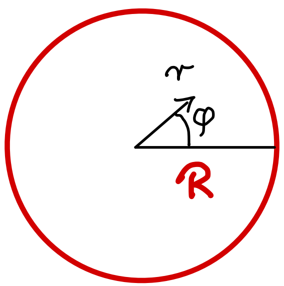


$n_x, n_y = 1, 2, \ldots$

b.c. ? ✓

· circular billiard

$$\psi_{k,l}(r,\varphi) = J_k\left(\frac{r}{R} j_{kl}\right) \cdot \begin{cases} \cos k\varphi & k=0,1,2\ldots \\ \sin k\varphi & k=1,2,\ldots \end{cases}$$



b.c. ?

$J_k(x) =$ k-th Bessel function of first kind

j_{kl} : l-th zero of $J_k(x)$: $J_k(j_{kl}) = 0$
$l = 1, 2, \ldots$

b) ergodic systems

- Random wave model (Berry 1977)

superposition of plane waves with random direction, phase, and amplitude

$$\psi(\vec{r}) = \sum_{n=1}^{N} a_n \cos(\vec{k}_n \vec{r} + \varphi_n)$$

- $N \to \infty$
- $\vec{k}_n$: $\frac{\hbar^2 \vec{k}_n^2}{2m} = E$ with random directions (uniform distribution)
- φ_n random phase (uniform distribution)
- $a_n \in \mathbb{R}$ (Gaussian distributed)

observation: "chaotic" eigenfunctions behave like RWM

Consequences of RWM:

- amplitude distribution $\psi(\vec{r})$: Gaussian

- spatial correlation $\langle \psi(\vec{r}+\vec{d})\,\psi(\vec{r}) \rangle_r = \frac{1}{A} J_0(k|\vec{d}|)$

  (follows already from restriction to energy shell)

- nodal patterns

- Quantum ergodicity theorem

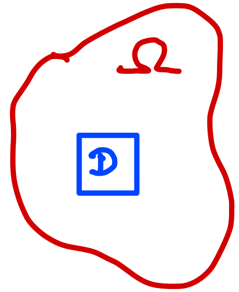
ergodic system: $\exists$ subsequence $\{n_j\}$ of density one s.t.

$$\lim_{j\to\infty} \langle \psi_{n_j} | A | \psi_{n_j} \rangle = \text{classical expectation value of operator } A$$

remarks:

- subsequence $\{n_j\}$ of density one: $\lim\limits_{E\to\infty} \frac{\#\{n_j \mid E_{n_j} < E\}}{N(E)} = 1$

- example for A: $A = P_D$ projection on region D in billiard



$$\lim_{j\to\infty} \langle \psi_{n_j} | P_D | \psi_{n_j} \rangle = \lim_{j\to\infty} \int_D |\psi_{n_j}(\vec{r})|^2 \overset{!}{=} \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)}$$

- exceptions of measure zero:
  - scars : enhancement of $\psi(\vec{r})$ close to unstable periodic orbit
  - bouncing ball modes: enhancement close to marginally stable p.o.

c) mixed systems (regular and chaotic)

$\rightarrow$ eigenfunctions in phase space needed