8.3. Phase space representation

Eigenfunction

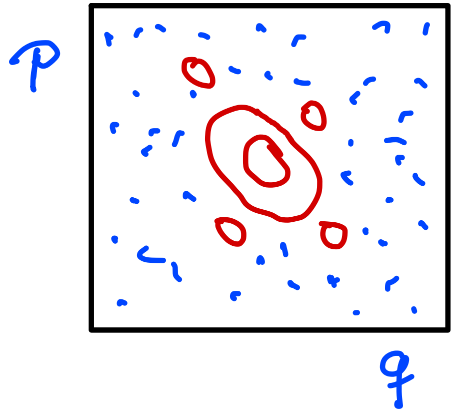
$\psi(q)$

$$\tilde{\psi}(p) = \int dq\, e^{-i\frac{pq}{\hbar}}\, \psi(q)$$

comparison?

$\longleftrightarrow$

Classical phase space of mixed system



naive: $\psi(q) \cdot \tilde{\psi}(p)$ ↯

Wigner function: $$W(q,p) = \frac{1}{h}\int_{-\infty}^{\infty} dx\; e^{i\frac{px}{\hbar}}\; \psi^*(q+\frac{x}{2})\,\psi(q-\frac{x}{2})$$

properties:

i) normalized $\int dp \int dq\; W(q,p) = 1$

ii) correct projection on q and p

$$\int dp\; W(q,p) = |\psi(q)|^2$$

$$\int dq\; W(q,p) = |\tilde{\psi}(p)|^2$$

iii) full information

iv) real, but can be negative $\Rightarrow$ no probability distribution

v) structures on sub-Planck cell scale: physical?

Husimi function : $H(q_0, p_0) = \frac{1}{h} \left| \langle \alpha(q_0,p_0) | \psi \rangle \right|^2$

Coherent state $|\alpha(q_0,p_0)\rangle$ centered at (q_0,p_0) :

$$\langle q | \alpha(q_0,p_0)\rangle = \frac{1}{N_q} \; e^{-\frac{(q-q_0)^2}{4(\Delta q)^2}} \; e^{i\frac{p_0 q}{\hbar}}$$

$$\langle p | \alpha(q_0,p_0)\rangle = \frac{1}{N_p} \; e^{-\frac{(p-p_0)^2}{4(\Delta p)^2}} \; e^{-i\frac{q_0(p-p_0)}{\hbar}}$$

minimal uncertainty : $\Delta p \cdot \Delta q = \frac{\hbar}{2}$

properties: i) real and positive

ii) "coarse grained" Wigner function (convolution by Gaussian)

iii) information lost

iv) no structures on sub-Planck cell scale

8.4. Phase space properties

8.4.1. Semiclassical eigenfunction hypothesis (Percival 1973, Berry 1977)

In the semiclassical limit
eigenfunctions concentrate on those regions in phase space which a typical orbit explores in the long-time limit $t \to \infty$.

regular eigenfunctions: localization on invariant tori

chaotic eigenfunctions: uniformly distributed on chaotic component with fluctuations on scale of Planck cell

8.4.2. Consequences for eigenvalue statistics

Generic Hamiltonian system $\Rightarrow$ mixed phase space

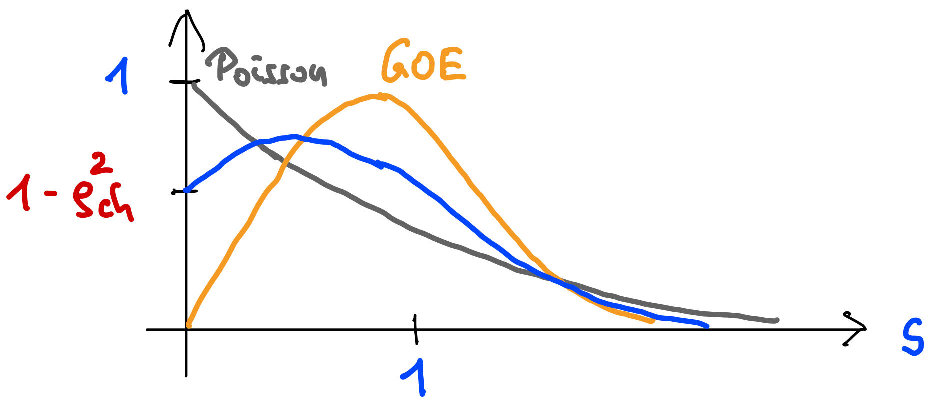
$\Rightarrow$ q.m.: regular <u>and</u> chaotic eigenfunctions

regular eigenfunctions $\Rightarrow$ Poisson spectrum
chaotic eigenfunctions $\Rightarrow$ GOE spectrum
} superimposed
assumption: independent

$$P(s) = \frac{d^2}{ds^2}\left[e^{-\varrho_{reg} s} \operatorname{erfc}\left(\frac{\sqrt{\pi}}{2} \varrho_{ch} s \right) \right]$$

Berry, Robnik 1984

phase space fraction



8.4.3. Advanced eigenfunction properties

- dynamical tunneling (classically forbidden, q.m. allowed)

$$H = \begin{pmatrix} E_r & v \\ v^* & E_c \end{pmatrix}$$

v: tunneling matrix element

regular-to-chaotic tunneling rate for many chaotic states:

Fermi's golden rule $\gamma_m = \frac{2\pi}{\hbar} |v_m|^2 \, d_{ch}$

γ_m: from torus m

d_{ch}: density of chaotic states

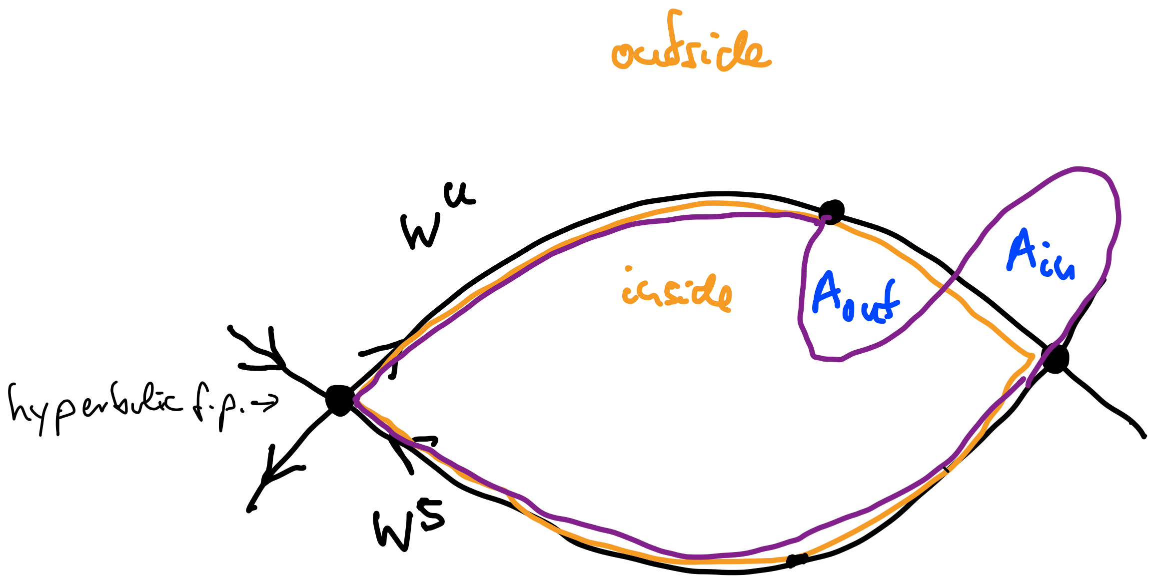
→ smallest from center of regular island

→ consequences for level statistics at small s: $P(s) \sim s^\alpha$ Mertig et al. 2011

α non-integer

→ amphibious states, flooding of islands, ...

• Partial barriers (cl. allowed transport, q.m. forbidden)



lobe area $> h$: open for q.m. ⇒ no consequences for eigenstates

lobe area $< h$: closed for q.m. ⇒ barrier for eigenstates and wave packet dynamics

→ hierarchical states:

chaotic region between island and part. barrier } ⇒ states:
- locally: chaotic
- globally: close to regular

$f_{hier} \sim h^{1-\frac{1}{\gamma}}$

if $P_{cl}(t) \sim t^{-\gamma}$