9. Time-periodic quantum systems

motivation: simplest chaotic Hamiltonian system:

1 d.o.f. + time-periodic driving

9.1. Floquet theory

Schrödinger eq.: $i\hbar \frac{\partial}{\partial t} \varphi(x,t) = H(t)\,\varphi(x,t)$ with $H(t+\underset{\text{period}}{T}) = H(t)$

- time-independent H: stationary states $\psi_n(x,t) = e^{-i\frac{E_n t}{\hbar}} u_n(x)$ with $H u_n = E_n u_n$

- Floquet theorem (first order diff. eq. with periodic coefficients)

$$\psi_n(x,t) = \underbrace{e^{-\frac{i}{\hbar}\varepsilon_n t}}_{\text{time-dep. phase}}\ \underbrace{u_n(x,t)}_{\text{time-periodic}} \quad \text{with } u_n(x,t+T) = u_n(x,t)$$

analogy to Bloch theorem:

$$H(x+a) = H(x) \Rightarrow \underset{\substack{\uparrow \\ \text{Bloch phase}}}{\psi_k(x)} = e^{ikx} u_k(x) \quad \text{with} \quad u_k(x+a) = u_k(x)$$

difference: k continuous, ε_n discrete from eigenvalue problem

remarks:

- general solution: $\varphi(x,t) = \sum_n a_n \psi_n(x,t)$
- $\psi_n(x,t)$ Floquet state
- ε_n quasienergy, defined up to additive constant $z\hbar\omega$ $\quad z \in \mathbb{Z}$, $\omega = \frac{2\pi}{T}$

$$e^{-\frac{i}{\hbar}\varepsilon_n t} u_n(x,t) = e^{-\frac{i}{\hbar}\overbrace{(\varepsilon_n + z\hbar\omega)}^{\tilde{\varepsilon}_n} t} \underbrace{e^{iz\omega t} u_n(x,t)}_{\tilde{u}_n(x,t) \text{ period } T}$$

no new solution

for uniqueness: $\varepsilon_n \in [0, \hbar\omega)$ or $\varepsilon_n \in [-\frac{\hbar\omega}{2}, \frac{\hbar\omega}{2})$

Floquet-operator (time-evolution operator over one period)

$$U(t+T,t) = \mathcal{T}\, e^{-\frac{i}{\hbar}\int_t^{t+T} dt'\, H(t')}$$

time-ordering operator

apply to arbitrary state: $U(t+T,t)\,\varphi(x,t) = \varphi(x,t+T)$

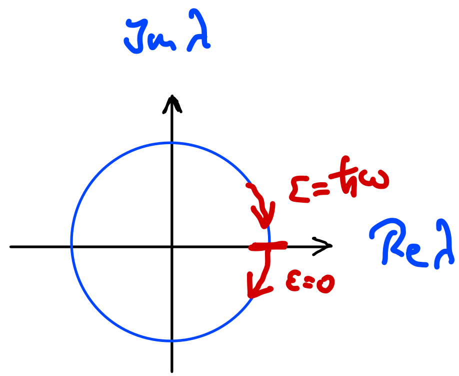
apply to Floquet state: $U(t+T,t)\,\psi_n(x,t) = \psi_n(x,t+T)$

$$= e^{-\frac{i}{\hbar}\varepsilon_n (t+T)}\, \underbrace{u_n(x,t+T)}_{=u_n(x,t)}$$

eigenstate of U

eigenvalue λ of U

$$= e^{-\frac{i}{\hbar}\varepsilon_n T}\, \psi_n(x,t)$$



eigenvalues λ of unitary operator U on unit circle

Approach to get $\psi_n(x,t)$:

1. choose $t=0$: determine $U(T,0)$ in some basis

2. solve eigenproblem: $U(T,0)\,\psi_n(x,t=0) = e^{-\frac{i}{\hbar}\varepsilon_n T}\,\psi_n(x,t=0)$

3. at other times $t = \tau + zT$; $0 \le \tau < T$, $z \in \mathbb{Z}$

$$\psi_n(x,t) = \underbrace{U(\tau+zT, zT)}_{U(\tau,0)}\, e^{-\frac{i}{\hbar}\varepsilon_n z T}\,\psi_n(x,t=0)$$

4. general solution: superposition

9.2. Quantum kicked rotor

Casati, Chirikov, Izraelev, Ford (1979)

motivation: · numerically simple (cl. + q.m.)

· all phenomena

9.2.1. Time-evolution operator

Hamiltonian with dimensions:

$$H(t) = \frac{L^2}{2J} + V\cos\Theta \, \underbrace{T\sum_{k=-\infty}^{\infty}\delta(t-kT)}_{\text{dimensionless}}$$

rotor: angle Θ

angular momentum L

moment of inertia J $(= mr^2)$

kick: time period T

strength V

q.m.: $[L,\Theta] = -i\hbar$

dimensionless variables:

$$\tilde{L} = \frac{L}{J/T} \longrightarrow p$$

$$\tilde{t} = \frac{t}{T} \longrightarrow t$$

$$\tilde{H} = \frac{H}{J/T^2} \longrightarrow H$$

$$\tilde{V} = \frac{V}{J/T^2} \longrightarrow K$$

$$\tilde{\hbar} = \frac{\hbar}{J/T} \longrightarrow \hbar \ (\hbar_{eff})$$

$$\Rightarrow \boxed{H(t) = \frac{p^2}{2} + K \cos\theta \sum_{k=-\infty}^{\infty} \delta(t-k)}$$

with $[p, \theta] = -i\hbar$

classical time-evolution (standard map)

$$\dot{\theta} = \frac{\partial H}{\partial p} = p$$

$$\dot{p} = -\frac{\partial H}{\partial \theta} = K \sin\theta \sum_{k=-\infty}^{\infty} \delta(t-k)$$

just after kick k : θ_k, p_k

just before kick $k+1$: $\theta_{k+1} = \theta_k + p_k$

just after kick $k+1$: $p_{k+1} = p_k + K \sin\theta_{k+1}$

q.m. time evolution: Floquet operator

$$U(1^+, 0^+) = U(1^+, 1^-)\, U(1^-, 0^+)$$

$$= e^{-\frac{i}{\hbar} K \cos\theta} \quad e^{-\frac{i}{\hbar}\frac{p^2}{2}}$$

remark: simple factorization of kinetic and potential term due to kick

9.2.2. Momentum basis

Choose periodic boundary condition in θ (in general: Bloch phase)

$$\varphi(\theta+2\pi) = \varphi(\theta)$$

Consequence for momentum representation: discrete momenta

$$\tilde{\varphi}(p) := \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\theta' \, e^{-\frac{i}{\hbar}p\theta'}\varphi(\theta')$$

$\theta' = 2\pi z + \theta \qquad z \in \mathbb{Z}, \; \theta \in [0, 2\pi)$

$$= \frac{1}{2\pi\hbar}\int_0^{2\pi} d\theta \underbrace{\sum_{z=-\infty}^{\infty} e^{-\frac{i}{\hbar}p(2\pi z+\theta)}}_{\sum_{n=-\infty}^{\infty}\delta\left(\frac{p}{\hbar}-n\right)} \underbrace{\varphi(2\pi z+\theta)}_{=\varphi(\theta)}$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{\hbar}\delta\left(\frac{p}{\hbar}-n\right)}_{\delta(p-n\hbar)} \underbrace{\frac{1}{2\pi}\int_0^{2\pi} d\theta \, e^{-in\theta}\varphi(\theta)}_{=:\varphi_n}$$

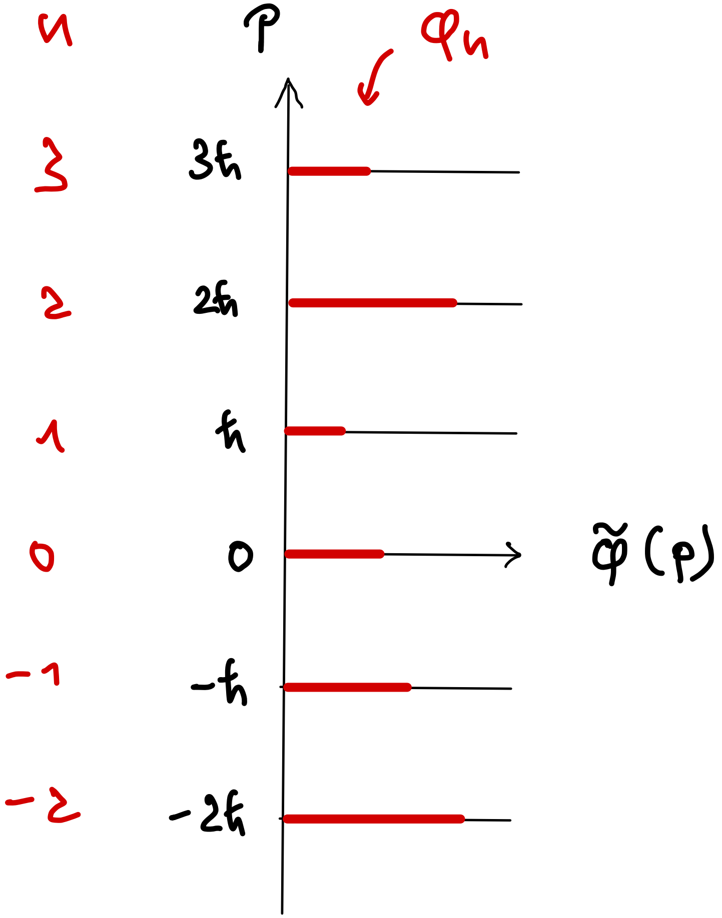
Dirac comb: Fourier series

$$\sum_{n=-\infty}^{\infty}\delta(x-n) = \sum_{z=-\infty}^{\infty} e^{-i2\pi x z}$$

$$\tilde{\varphi}(p) = \sum_{n=-\infty}^{\infty} \varphi_n \, \delta(p - n\hbar)$$

just discrete momenta $p = n\hbar$ with $n \in \mathbb{Z}$

wave function in discrete momentum basis $|n\rangle$



example: $\varphi(\theta) = e^{im\theta} = \langle \theta | m \rangle$

$$\varphi_n = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{-in\theta} e^{im\theta} = \delta_{n,m}$$

momentum basis state $|m\rangle$

general: $\varphi(\theta) = \langle \theta | \varphi \rangle$

$\varphi_n = \langle n | \varphi \rangle$

$$\mathbb{1} = \sum_{n=-\infty}^{\infty} |n\rangle\langle n|$$

$$\mathbb{1} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, |\theta\rangle\langle\theta|$$

Floquet operator in discrete momentum basis

$$|\varphi(t+1)\rangle = \underbrace{U(1^+, 0^+)}_{e^{-\frac{i}{\hbar}K\cos\hat\theta}\; e^{-\frac{i}{\hbar}\frac{\hat p^2}{2}}} |\varphi(t)\rangle \qquad t = 0^+$$

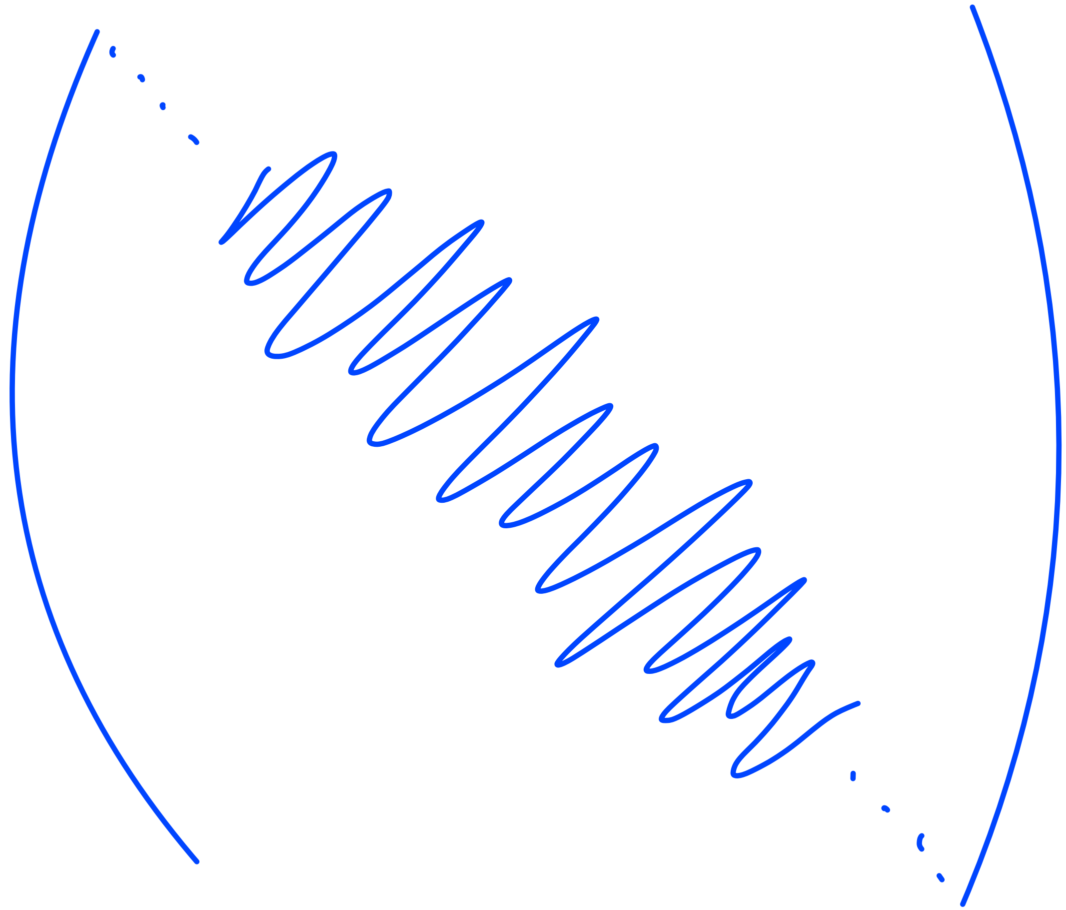
$\langle n|$

$$\mathbb{1} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, |\theta\rangle\langle\theta| \qquad \mathbb{1} = \sum_{m=-\infty}^{\infty} |m\rangle\langle m|$$

$$\varphi_n(1^+) = \sum_{m=-\infty}^{\infty} \underbrace{\frac{1}{2\pi}\int_0^{2\pi} d\theta\; e^{-\frac{i}{\hbar}K\cos\theta} \underbrace{\langle n|\theta\rangle\langle\theta|m\rangle}_{e^{i(m-n)\theta}} \underbrace{e^{-\frac{i}{\hbar}\frac{(m\hbar)^2}{2}}}_{e^{-i\hbar\frac{m^2}{2}}}}_{=:\, U_{nm}} \varphi_m(0^+)$$

$$\varphi_n(1^+) = \sum_{m=-\infty}^{\infty} U_{nm}\, \varphi_m(0^+) \qquad U_{nm} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\; e^{-\frac{i}{\hbar}K\cos\theta}\, e^{i(m-n)\theta}\, e^{-i\hbar\frac{m^2}{2}}$$

U_{nm} decays exponentially with $m-n \gg 1$

$U_{nm} =$ 

Time-evolution operator on cylinder:

infinite matrix U_{nm} $\Rightarrow$ numerically not usable

(subblocks work in some cases)

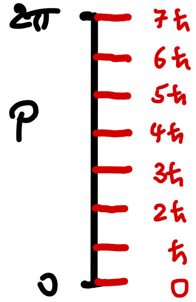
Time-evolution operator on torus $[0,2\pi) \times [0,2\pi)$

periodic boundary condition in p-direction

after one unit cell: $\tilde{\varphi}(p+2\pi) = \tilde{\varphi}(p) = \sum_{n=-\infty}^{\infty} \varphi_n \, \delta(p - n\hbar)$

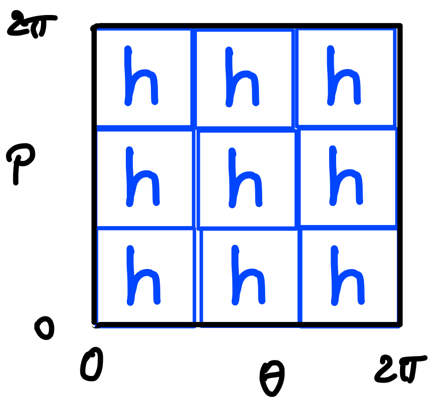
$$\Rightarrow \quad 2\pi \overset{!}{=} N\hbar \quad \Rightarrow \quad \boxed{\hbar = \frac{2\pi}{N}} \qquad \text{with } N \in \mathbb{N}$$

quantization on torus for these $\hbar$ possible



- discrete momentum basis has period 2π

- equivalent: integer number of Planck cells h on torus



$$(2\pi)^2 \overset{!}{=} N h \quad \checkmark$$

aim: infinite U_{nm} $\longrightarrow$ $N \times N$ matrix U^N_{nm} (not $N \times N$ subblock)

p-periodic wave functions: $\varphi_{n+zN} = \varphi_n$, $z \in \mathbb{Z}$

$$\varphi_n(1^+) = \sum_{m'=-\infty}^{\infty} U_{nm'} \varphi_{m'}(0^+)$$

$m' = m + zN$

$m = 0, 1, \dots, N-1$
$z \in \mathbb{Z}$

$$= \underbrace{\sum_{m=0}^{N-1} \sum_{z=-\infty}^{\infty} U_{n,m+zN}}_{U^N_{nm}} \underbrace{\varphi_{m+zN}(0^+)}_{\varphi_m(0^+)}$$

$$U^N_{nm} = \sum_{z=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{-\frac{i}{\hbar} K \cos\theta} \, e^{i(m+zN-n)\theta} \, e^{-i\hbar \frac{(m+zN)^2}{2}}$$

- $$e^{-i\hbar\frac{(m+zN)^2}{2}} \overset{\hbar=\frac{2\pi}{N}}{=} e^{-i\pi\frac{(m+zN)^2}{N}} = e^{-i\pi\frac{m^2}{N}}\, \underbrace{e^{-i\pi 2mz}}_{=1}\, \underbrace{e^{-i\pi z^2 N}}_{=1 \text{ for } \underline{N \text{ even}}} = e^{-i\pi\frac{m^2}{N}}$$

- $$\sum_{z=-\infty}^{\infty} e^{izN\theta} = \sum_{k=-\infty}^{\infty}\delta\left(\frac{N\theta}{2\pi}-k\right) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\theta-\frac{2\pi k}{N}\right)$$

Dirac comb: Fourier series

$$\sum_{n=-\infty}^{\infty}\delta(x-n) = \sum_{z=-\infty}^{\infty} e^{-i2\pi x z}$$

$$\Rightarrow U^N_{nm} = \frac{1}{N}\sum_{k=-\infty}^{\infty}\int_0^{2\pi} d\theta\, \delta\left(\theta-\frac{2\pi k}{N}\right) e^{-\frac{i}{\hbar}K\cos\theta}\, e^{i(m-n)\theta}\, e^{-i\pi\frac{m^2}{N}}$$

N even

$$\Rightarrow \boxed{U^N_{nm} = \frac{1}{N}\sum_{k=0}^{N-1} e^{-i\frac{N}{2\pi}K\cos\frac{2\pi k}{N}}\, e^{i2\pi\frac{(m-n)k}{N}}\, e^{-i\pi\frac{m^2}{N}}}$$

Chang, Shi PRA 34, 7 (1986)

9.2.3. Numerical considerations

- matrix elements:
  - naively: N^2 elements, each N-fold sum $\Rightarrow \propto N^3$ operations
  - better: $\propto N^2$ operations

$$\boxed{\text{N even} \quad U^N_{nm} = \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{\underbrace{e^{-i \frac{N}{2\pi} K \cos \frac{2\pi k}{N}}}_{V_k} \, e^{i 2\pi \frac{(m-n)k}{N}}}_{f_{m-n}} \, \underbrace{e^{-i\pi \frac{m^2}{N}}}_{T_m}}$$

$$U^N_{nm} = f_{m-n} T_m \qquad f_\ell = \frac{1}{N} \sum_{k=0}^{N-1} V_k \, e^{i 2\pi \frac{\ell k}{N}}$$

- time evolution

a) matrix-vector multiplication

$$\varphi_n(t+1) = \sum_{m=0}^{N-1} U^N_{nm}\, \varphi_m(t) \qquad \propto N^2$$

b) using discrete Fourier transformation

$$\varphi_n(t+1) = \sum_{m=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{e^{-i\frac{N}{2\pi}K\cos\frac{2\pi k}{N}}}_{V_k} \; e^{i2\pi\frac{(m-n)k}{N}} \; \underbrace{e^{-i\pi\frac{m^2}{N}}}_{T_m} \; \varphi_m(t)$$

$$= \underbrace{\frac{1}{N} \sum_{k=0}^{N-1} e^{-i2\pi\frac{nk}{N}}}_{\text{inverse FT}} V_k \underbrace{\sum_{m=0}^{N-1} e^{i2\pi\frac{mk}{N}}}_{\text{discrete FT}} T_m\, \varphi_m(t) \qquad \propto N \log N$$

- eigenstates

$$\mathcal{U}(T,0)\,\psi(x,t=0) = e^{-\frac{i}{\hbar}\varepsilon T}\,\psi(x,t=0)$$

- diagonalize $N \times N$ matrix $\mathcal{U}^N_{nm}$

  $\Rightarrow$ quasienergies ε

  Floquet states ψ

- semiclassical limit: $\hbar = \frac{2\pi}{N} \longrightarrow 0 \iff N \longrightarrow \infty$

- number of eigenstates: $N = \dfrac{\text{phase space area}}{\text{Planck cell}} = \dfrac{(2\pi)^2}{h}$

9.2.4. Comparison with classical phase space

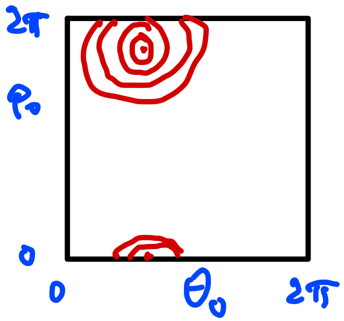
Husimi function : $H(\theta_0, p_0) = \frac{1}{h} \left| \langle \alpha(\theta_0, p_0) | \psi \rangle \right|^2 = \frac{1}{h} \left| \sum_{n=0}^{N-1} \psi_n^* \, \alpha_n(\theta_0, p_0) \right|^2$

Coherent state $|\alpha(\theta_0, p_0)\rangle$ centered at (θ_0, p_0) :

$$\alpha_n(\theta_0, p_0) = \langle n\hbar | \alpha(\theta_0, p_0) \rangle \quad \propto \quad e^{-\frac{(n\hbar - p_0)^2}{4(\Delta p)^2}} \; e^{-i \frac{\theta_0 (n\hbar - p_0)}{\hbar}}$$

choose $\Delta p = \Delta \theta = \sqrt{\frac{\hbar}{2}} = \sqrt{\frac{\pi}{N}}$

periodicity in p_0 :

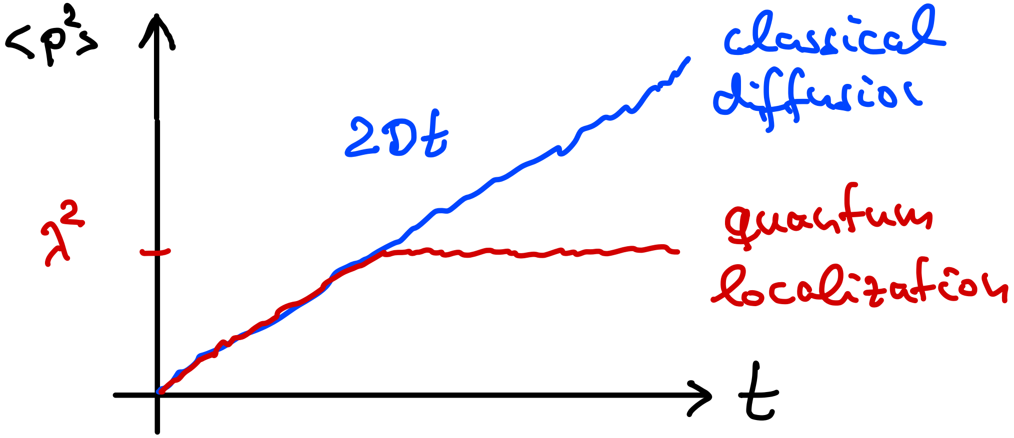


$$\sum_{z=-\infty}^{\infty} \alpha_n \left(\theta_0, p_0 + z \cdot 2\pi \right)$$

$$\Rightarrow \quad H(\theta_0, p_0) \propto \left| \sum_{z=-1}^{1} \sum_{n=0}^{N-1} \psi_n^* \, e^{-\frac{\pi}{N}\left(n - \frac{p_0 N}{2\pi} - zN\right)^2} e^{-i\theta_0\left(n - \frac{p_0 N}{2\pi} - zN\right)} \right|^2$$

9.3. Dynamical Localization

Chirikov, Izrailev, Shepelyansky (1981)



- quantum interference effect (no simple intuition)
- occurs if classical chaotic dynamics leads to diffusion
- cl. allowed transport, q.m. forbidden
- related to Anderson localization in disorder potential (Maryland transformation) Fishman, Grempel, Prange 1982

Kicked rotor on cylinder: localization in momentum space

$K \gg 1$: classical diffusion $\langle p^2(t) \rangle = 2Dt$ with $D \approx \frac{K^2}{4}$

$\hbar \neq \frac{2\pi}{N}$: momentum grid $p = n\hbar$ incommensurate to period 2π

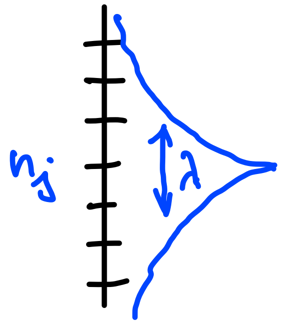
What is localization length?

$$\lambda = \lambda_{grid}\,\hbar$$

in units of p — in momentum grid

- Assume eigenfunction $|\psi_j\rangle$ localizes at grid point n_j, all with identical loc. length λ_{grid}

$$|\langle n|\psi_j\rangle|^2 \propto e^{-\frac{|n-n_j|}{\lambda_{grid}}}$$



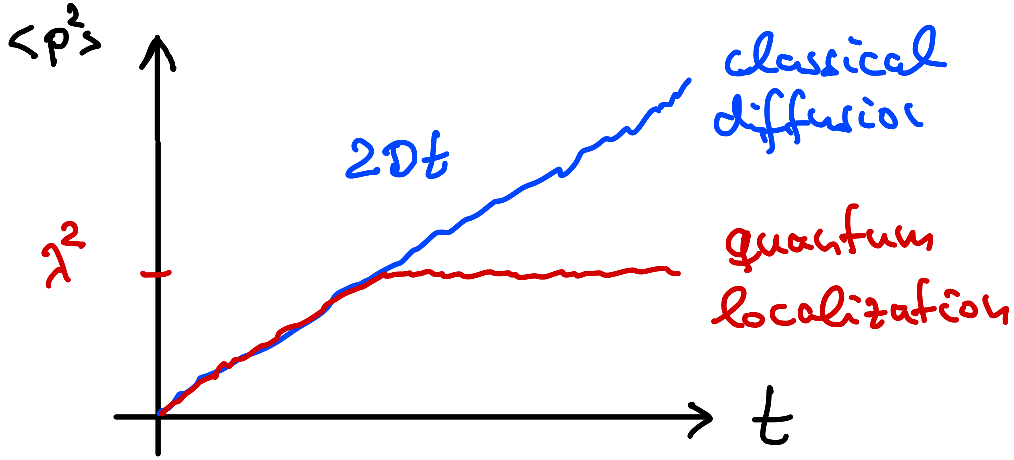
$\Rightarrow$ Initial local wave packet excites $\sim \lambda_{grid}$ different eigenstates

$$|\varphi(t)\rangle = \sum_j \langle\psi_j|\varphi(t=0)\rangle\, e^{-i\varepsilon_j t}\,|\psi_j\rangle$$

relevant if n_j near $\varphi(t=0)$

Siberian argument

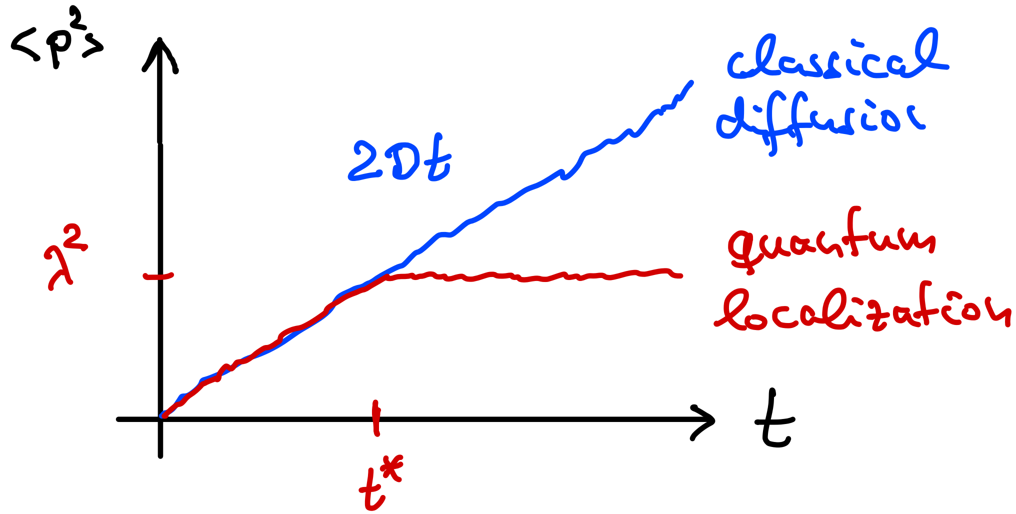
Chirikov, Izrailev, Shepelyansky (1981)



$\Rightarrow$ relevant quasienergies ε_j have typical distance: $\Delta\varepsilon = \dfrac{2\pi}{\lambda_{grid}}$

$\Rightarrow$ effective Heisenberg time: $t^* = \dfrac{2\pi}{\Delta\varepsilon} = \lambda_{grid} = \dfrac{\lambda}{\hbar}$

$\Rightarrow$ $2D\dfrac{\lambda}{\hbar} = 2Dt^* \overset{!}{=} \lambda^2$



$$\Rightarrow \boxed{\lambda = \frac{2D}{\hbar}}$$

remark: $\hbar \to 0$: $\lambda \to \infty$

$\lambda_{grid} \to \infty$

$t^* \to \infty$ quantum follows cl. diffusion for longer times