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Chaos and Quantum Chaos

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3. Exercise - Solutions

3.1 Wanted: Dynamical system

The powers of 2 are given by: $1, 2, 4, 8, 16, 32, 64, \ldots$
Taking the first digit gives the infinite sequence: $1, 2, 4, 8, 1, 3, 6, \ldots$
Answer the following questions:

1. Will the digit 7 occur in this sequence?
2. Will it occur as often as 8?
3. What is the probability for each digit to occur?

Hint: Find a dynamical system on a 1D torus, which generates the sequence $1, 2, 4, 8, 1.6, 3.2, 6.4, \ldots$.

Solution:
Initial iteration: $x_n \in \mathbb{R}$

$$x_{n+1} = 2\,x_n$$

Iteration reduced to interval for first digit: $y_n \in [1, 10)$

$$y_{n+1} = 10^{\log_{10}(2\,y_n) \mod 1}$$

Define logarithmic variable: $z_n \in [0, 1)$

$$z_n = \log_{10}(y_n)$$

Dynamical system:

$$\begin{aligned} z_{n+1} &= z_n + \log_{10}(2) \mod 1 \\ \text{solution:} \qquad z_n &= z_0 + n\,\log_{10}(2) \mod 1 \end{aligned}$$

Integrable dynamics on 1D-torus:
$\log_{10}(2)$ is irrational $\Rightarrow$ irrational torus
$\Rightarrow$ ergodic dynamics for z_n (time-average equals state space average)
In integrable systems this holds for any initial condition (z_0), see proof in lecture and exercise 2.2.
Remark: In general systems with ergodic dynamics (e.g. chaotic systems), one typically has initial conditions which are exceptions of measure zero, in particular initial conditions on the stable invariant manifold of a hyperbolic fixed point.

Ergodicity implies that the probability for a first digit k is given by the corresponding fraction of state space:

$$P(k) = \frac{\log_{10}(k+1) - \log_{10}(k)}{\log_{10}(10) - \log_1(k)} = \log_{10}(1 + \frac{1}{k})$$

Examples (up to third digit):

$$\begin{aligned} P(1) &= 0.301 \\ P(7) &= 0.058 \\ P(8) &= 0.051 \end{aligned}$$