SS2019: Solid State Theory
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# Problem Set 2

1. **Free electron gas in low dimensions** (2+2+2+2+2 points)

Consider free electron gases confined in 1D and 2D boxes with "volume" $V$. The total number of electrons is $N_e$.

(a) Calculate the density of states (DOS) $D(\varepsilon)$, plot them and compare with the 3D result.

(b) Determine the Fermi energies $\varepsilon_{\rm F}$ by using the DOS obtained in (a). What are the corresponding Fermi wave vectors $k_{\rm F}$?

(c) Calculate the total energies of the Fermi gas at zero temperature. Compare your results with the 3D case ($E_{\rm 3D}/N_e = \frac{3}{5}\varepsilon_{\rm F}$).

(d) Now consider the low temperature regime $T \ll T_{\rm F} = \varepsilon_{\rm F}/k_{\rm B}$ and calculate the internal energies of the 1D and 2D free electron gases

$$U(T) = \int_{-\infty}^{\infty} d\varepsilon\, \varepsilon D(\varepsilon) f(\varepsilon, \mu, T)\,,$$

where $f(\varepsilon, \mu, T) = 1/(e^{\beta(\varepsilon-\mu)}+1)$ is the Fermi-Dirac distribution function.

Hint: You may use the Sommerfeld expansion to approximate the integral.

(e) Calculate the specific heat based on the results obtained in (d). How does the specific heat behave when $T \to 0$?

2. **Density-density correlation function** (3+3+2 points)

Consider the Fermi sphere of a *spinless* Fermi gas in 3D, described by the wave function

$$|{\rm FS}\rangle = \prod_{|\mathbf{k}|<k_{\rm F}} c_{\mathbf{k}}^{\dagger}|0\rangle\,.$$

(a) Calculate the Green's function

$$G(\mathbf{r}_1, \mathbf{r}_2) = \langle \psi^{\dagger}(\mathbf{r}_1)\psi(\mathbf{r}_2)\rangle\,,$$

where $\psi(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} c_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$ and $\langle\cdots\rangle$ is the expectation value with respect to $|{\rm FS}\rangle$. We note that $G(\mathbf{r}_1, \mathbf{r}_2)$ is often referred to as the one-particle reduced density matrix or correlation matrix.

(b) Calculate the density-density correlation function

$$C(\mathbf{r}_1, \mathbf{r}_2) = \langle n(\mathbf{r}_1)n(\mathbf{r}_2)\rangle - \langle n(\mathbf{r}_1)\rangle\langle n(\mathbf{r}_2)\rangle\,,$$

where $n(\mathbf{r}) = \psi^{\dagger}(\mathbf{r})\psi(\mathbf{r})$ is the particle-density operator.

Hint: For calculating $\langle n(\mathbf{r}_1)n(\mathbf{r}_2)\rangle$, you may switch to momentum space or use the Wick's theorem $\langle \psi^{\dagger}(\mathbf{r}_1)\psi(\mathbf{r}_1)\psi^{\dagger}(\mathbf{r}_2)\psi(\mathbf{r}_2)\rangle = \langle \psi^{\dagger}(\mathbf{r}_1)\psi(\mathbf{r}_1)\rangle\langle \psi^{\dagger}(\mathbf{r}_2)\psi(\mathbf{r}_2)\rangle + \langle \psi^{\dagger}(\mathbf{r}_1)\psi(\mathbf{r}_2)\rangle\langle \psi(\mathbf{r}_1)\psi^{\dagger}(\mathbf{r}_2)\rangle$ .

(c) Plot $C(\mathbf{r}_1, \mathbf{r}_2)$ as a function of the distance $|\mathbf{r}_1 - \mathbf{r}_2|$ and interpret the results (e.g. short/large distance behaviors and a characteristic oscillation known as "Friedel oscillation").