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Problem Set 5

1. **Landau levels in symmetric gauge** (2+2+2+2 points)

In the symmetric gauge, the Hamiltonian for an electron moving in two dimensions (x-y plane) under a perpendicular magnetic field $\vec{B} = B\hat{z}$ is given by

$$H = \frac{1}{2m}(p_x + eA_x)^2 + \frac{1}{2m}(p_y + eA_y)^2,$$

where $\vec{A} = (A_x, A_y, 0) = (-\frac{B}{2}y, \frac{B}{2}x, 0)$. For convenience, one may set $m = e = B = \hbar = 1$.

(a) Introduce complex coordinates $z = x - iy$ and $\bar{z} = x + iy$. Show that the Hamiltonian can be rewritten as

$$H = \frac{1}{2}\left(-4\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{4}z\bar{z} - z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right).$$

(b) Introduce bosonic operators

$$a = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right),$$
$$b = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right).$$

Show that these bosonic operators indeed satisfy the standard commutation relations, i.e., $[a, a^\dagger] = [b, b^\dagger] = 1$ (other commutators are zero).

(c) Show that the Hamiltonian is simply written as

$$H = a^\dagger a + \frac{1}{2},$$

and the eigenvalues/eigenvectors of the Landau levels are given by

$$H|n, m\rangle = E_n|n, m\rangle,$$

where $E_n = n + \frac{1}{2}$ and $|n, m\rangle = \frac{(b^\dagger)^{m+n}}{\sqrt{(m+n)!}}\frac{(a^\dagger)^n}{\sqrt{n!}}|0, 0\rangle$ $(m = -n, -n+1, \ldots)$.

(d) Show that the single-electron wave functions in the lowest Landau level take the following form in coordinate space:

$$\phi_{0,m}(z, \bar{z}) = \langle \mathbf{r}|0, m\rangle \propto z^m e^{-|z|^2/4}.$$

You are encouraged to derive the normalization constant (but not required to do so).

Ref: See Chapter 3 in the book “Composite Fermions” by J. K. Jain.

2. **Chiral edge states** (3+3+3 points)

The following Hamiltonian describes the quantum Hall system discretized on a square lattice:

$$H = t\sum_{m=1}^{N_x}\sum_{n=1}^{N_y}(e^{-in\alpha}c^\dagger_{m,n}c_{m+1,n} + e^{in\alpha}c^\dagger_{m+1,n}c_{m,n}) + t\sum_{m=1}^{N_x}\sum_{n=1}^{N_y-1}(c^\dagger_{m,n}c_{m,n+1} + c^\dagger_{m,n+1}c_{m,n}),$$

where $c^\dagger_{m,n}$ ($c_{m,n}$) is the electron creation (annihilation) operator at site $\mathbf{r} = (ma, na)$ (a: lattice spacing). The boundary condition is periodic in x-direction with $c_{m+N_x,n} = c_{m,n}$ and open in y-direction.

(a) Show that the Hamiltonian is reduced to

$$H = \sum_{k_x} \sum_{n,n'=1}^{N_y} c^\dagger_{k_x,n} [\mathcal{H}(k_x)]_{n,n'} c_{k_x,n'}$$

with

$$[\mathcal{H}(k_x)]_{n,n'} = t(\delta_{n',n+1} + \delta_{n',n-1}) + 2t\cos(k_x a - n\alpha)\delta_{n',n}$$

by using the Fourier transformation $c_{m,n} = \frac{1}{\sqrt{N_x}} \sum_{k_x} c_{k_x,n} e^{ik_x ma}$, where $k_x = 0, \pm\frac{2\pi}{N_x a}, \ldots, \frac{\pi}{a}$.

(b) Consider a finite-size lattice with $N_x = 100$, $N_y = 30$ and $a = 1$. The parameters in the Hamiltonian may be taken as $t = 5$ and $\alpha = 0.2$. Diagonalize the matrix $\mathcal{H}(k_x)$ numerically for each k_x to determine the single-particle energies $E_q(k_x)$,

$$\sum_{n,n'=1}^{N_y} [U(k_x)]_{q,n} [\mathcal{H}(k_x)]_{n,n'} [U(k_x)]^\dagger_{n',q'} = E_q(k_x)\delta_{q,q'},$$

where $U(k_x)$ is the unitary matrix encoding the first-quantized single-electron wave function (why?). Plot $E_q(k_x)$ as a function of k_x.

Hint: The Hamiltonian is diagonalized as

$$H = \sum_{k_x} \sum_{q=1}^{N_y} E_q(k_x) d^\dagger_{k_x,q} d_{k_x,q},$$

where $d_{k_x,q} = \sum_{n=1}^{N_y} [U(k_x)]_{q,n} c_{k_x,n}$.

(c) Pick up several single-electron wave functions (labeled by k_x and q) in the lowest Landau level. Can you distinguish which ones correspond to chiral edge states and which ones are "bulk" states? Verify your results by plotting $|[U(k_x)]_{q,n}|^2$ (probability of finding the electron at the n-th row of the lattice) as a function of n.