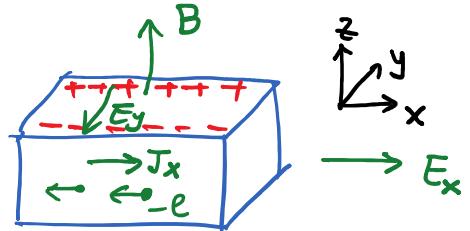


§ 5. Quantum Hall effect

*) 2D electron gas in a strong magnetic field

- Classical Hall effect



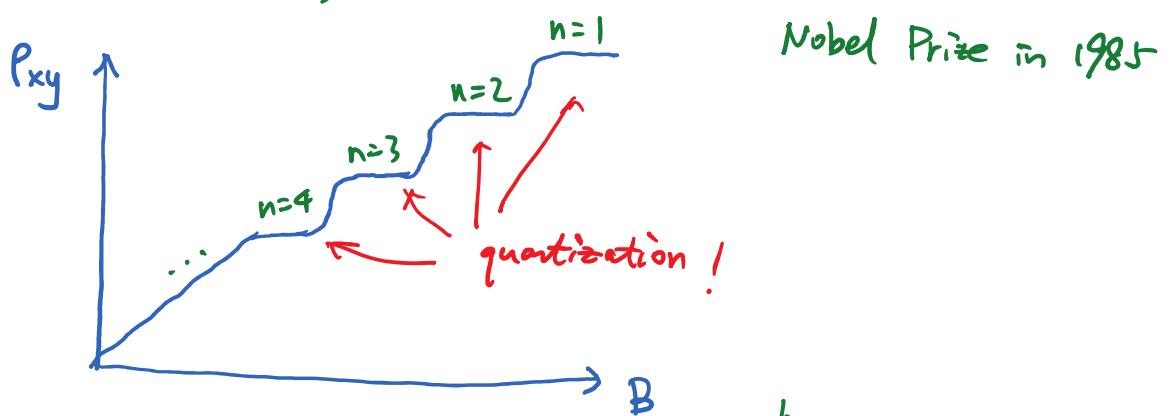
$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix} \stackrel{\approx 0}{\Rightarrow} \begin{cases} E_x = \rho_{xx} J_x \\ E_y = \rho_{yx} J_x \end{cases} \stackrel{\approx -\rho_{xy}}{=} -\rho_{xy}$$

Magnetoresistance: $\rho_{xx} = \frac{E_x}{J_x} = \frac{m}{ne^2 \tau}$

Hall coefficient:

$$R_H = \frac{1}{B} \rho_{yx} = \frac{1}{B} \frac{E_y}{J_x} = -\frac{1}{ne} \Rightarrow \rho_{xy} \propto B$$

- Integer quantum Hall effect (K. von Klitzing et al., 1980)



$$\rho_{xy} = \frac{1}{n} \frac{h}{e^2} \quad (n=1, 2, 3, \dots)$$

$$\frac{h}{e^2} = 25812,807 \Omega$$

(resistance standard)

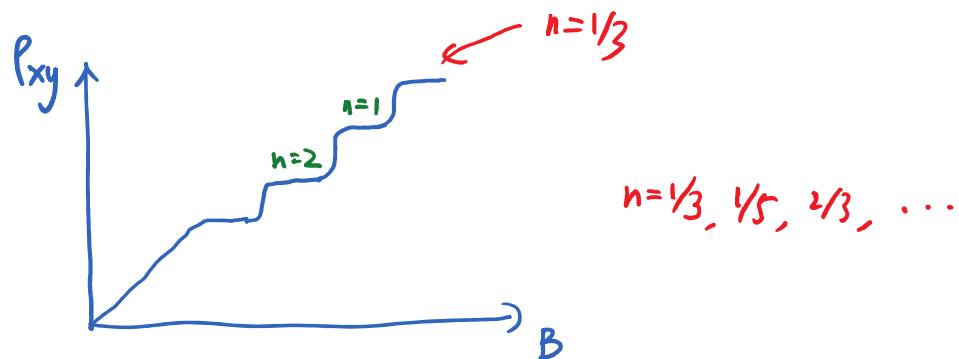
ρ_{xx} vanishingly small when ρ_{xy} quantized

- Fractional quantum Hall effect (FQH)

Experiment: D.C. Tsui, H.L. Stormer, A.C. Gossard, 1982

Theory: R.B. Laughlin, 1983

Nobel Prize in 1998



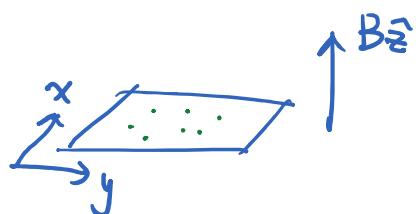
- Single-particle states and Landau levels

2D spin-polarized electron gas in strong B-field:

$$H = \frac{1}{2m} (\vec{p}_x + e\vec{A}_x)^2 + \frac{1}{2m} (\vec{p}_y + e\vec{A}_y)^2$$

effective mass vector potential

$$\vec{\nabla} \times \vec{A} = \vec{B} = B\hat{z}$$



Landau gauge: $\vec{A}(\vec{r}) = (A_x, A_y, A_z) = (-By, 0, 0)$

$$B_z = \partial_x A_y - \partial_y A_x = -\partial_y (-By) = B$$

Other gauge choices are possible, e.g.

$$\text{symmetric gauge: } \vec{A} = \left(-\frac{B}{2}y, \frac{B}{2}x, 0 \right)$$

Below we use the Landau gauge $\vec{A} = (-By, 0, 0)$:

$$H = \frac{1}{2m} (\hat{P}_x - eB\hat{y})^2 + \frac{1}{2m} \hat{P}_y^2$$

$$[H, \hat{P}_x] = 0 \quad [\hat{x}, \hat{P}_x] = [\hat{y}, \hat{P}_y] = i\hbar$$

→ conserved quantity

$$[H, \hat{P}_y] \neq 0$$

$$\Rightarrow H = \underbrace{\frac{1}{2m} (\underbrace{k_x \hat{x}}_{\downarrow \text{number}} - eB\hat{y})^2}_{\text{instead of operator}} + \frac{1}{2m} \hat{P}_y^2$$

number (instead of operator): $k_x = 0, \pm \frac{2\pi}{L_x}, \pm \frac{4\pi}{L_x}, \dots$

$$= \frac{1}{2m} \hat{P}_y^2 + \frac{1}{2} m \omega_c^2 (\hat{y} - \underbrace{y_0}_{\text{in x-direction}})^2 \quad (\text{PBC with length } L_x \text{ in x-direction})$$

$$\omega_c = \frac{eB}{m}$$

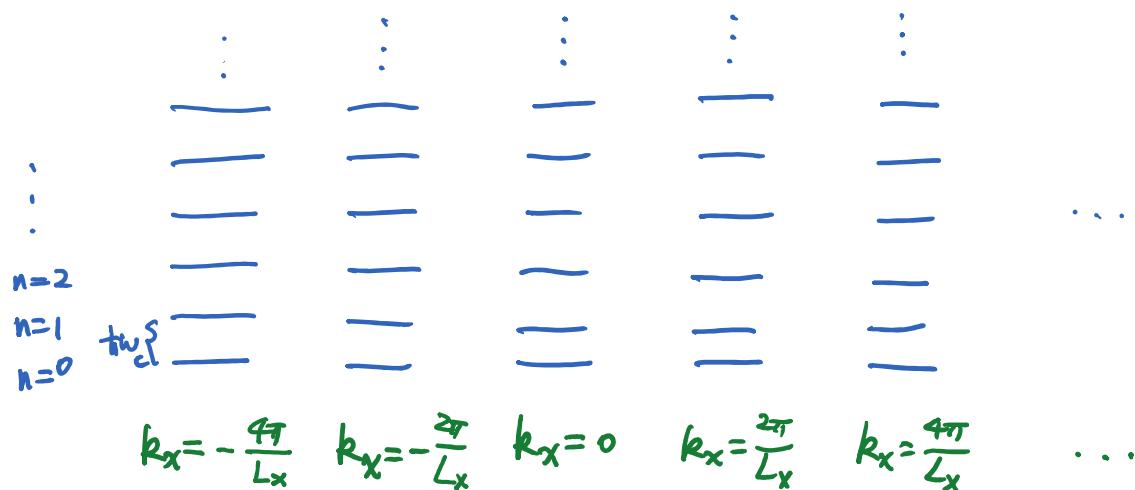
$$y_0 = \frac{\hbar}{eB} k_x$$

(cyclotron frequency)

Harmonic oscillator for each $y_0 = \frac{\hbar}{eB} k_x$!

Single-particle energies:

$$E_{n,k_x} = \left(n + \frac{1}{2}\right) \hbar \omega_c$$



- 1) E_{n,k_x} does NOT depend on k_x .
 ⇒ huge degeneracy!

- 2) Equally spaced energy levels (Landau levels)

$n=0$: lowest Landau level (LLL)

$n=1$: First Landau level

⋮

Many-electron states:

occupation of single-particle levels (which ones?)

Single-electron wave function:

$$\phi_{n, k_x}(\vec{r}) = \underbrace{\frac{1}{\sqrt{L_x}} e^{ik_x x}}_{\text{plane wave in } x\text{-direction}} \left(\frac{m\omega_c}{\pi\hbar} \right) \frac{1}{\sqrt{n!}} \underbrace{e^{-\frac{m\omega_c}{2\hbar}(y-y_0)^2}}_{H_n[(y-y_0)\sqrt{\frac{m\omega_c}{\hbar}}]} H_n[(y-y_0)\sqrt{\frac{m\omega_c}{\hbar}}]$$

↓
Hermite polynomial
 $H_0(t) = 1$
 $H_1(t) = t$
⋮

ϕ_{n, k_x} exponentially localized around

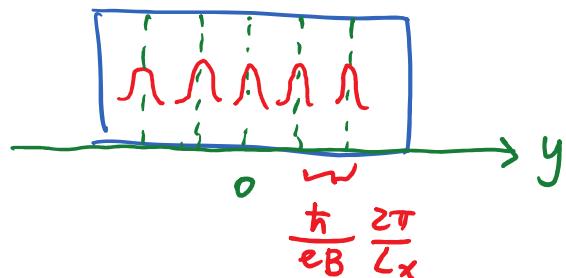
$$y = y_0 = \frac{\hbar}{eB} k_x = \frac{\hbar}{eB} \frac{2\pi}{L_x} n_x \quad (n_x = 0, \pm 1, \pm 2, \dots)$$

LLL ($n=0$): $\phi_{0, k_x} \sim e^{ik_x x} e^{-\frac{(y-y_0)^2}{2l^2}}$

↓
Gaussian wave packet

magnetic length $l = \sqrt{\frac{\hbar}{m\omega_c}} = \sqrt{\frac{\hbar}{m} \frac{m}{eB}} = \sqrt{\frac{\hbar}{eB}}$

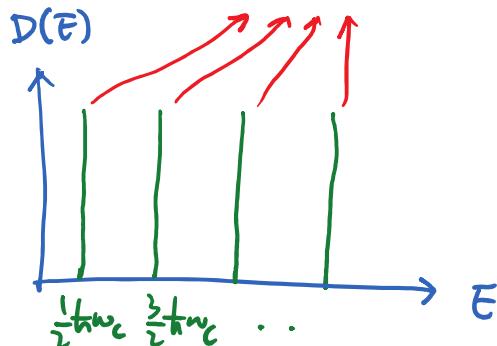
$l \sim 10^{-8} \text{ m} \gg \text{lattice spacing} \quad (\text{lattice ignored})$



Open boundary condition (OBC) with length L_y in y -direction \Rightarrow count the degeneracy!

Density of states:

degeneracy of each Landau level: $N_L(B)$



$$N_L(B) = \frac{L_y}{\frac{\hbar}{eB} \frac{2\pi}{L_x}} = \frac{eBL_xL_y}{\hbar} = \frac{e}{\hbar} BS$$

length in y-direction

distance between wave packets

$S = L_x L_y$
(total area)

$$= \frac{\phi(B)}{\phi_0} \rightarrow \text{flux } \phi = BS$$

unit of flux quantum

$$\phi_0 = \frac{\hbar}{e}$$

of electrons: N

filling fraction: $\nu = \frac{N}{\phi/\phi_0}$ (tunable by B)

e.g. $\nu = 1 \Rightarrow$ LLL fully occupied

$B \sim 10 T$ (each electron has one flux quantum)