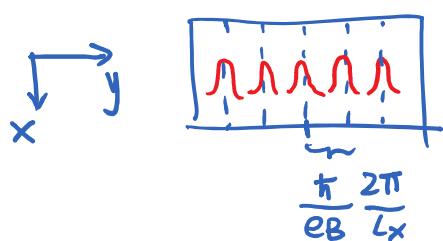
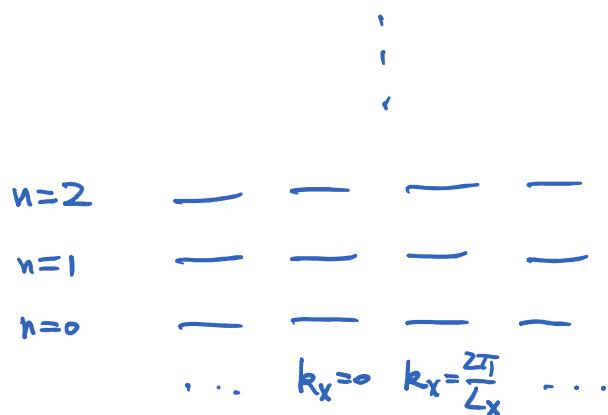


§5. Quantum Hall effect

2D electron gas in a strong magnetic field:

$$H = \frac{1}{2m} (\hat{p}_x - eB\hat{y})^2 + \frac{1}{2m}\hat{p}_y^2$$

$$E_{n,k_x} = (n + \frac{1}{2})\hbar\omega_c \quad \omega_c = \frac{eB}{m}$$



degeneracy:

$$N_L(B) = \frac{L_y}{\frac{\pi}{eB} \frac{2\pi}{L_x}} = \frac{BL_x L_y}{h/e} = \frac{\phi}{\phi_0}$$

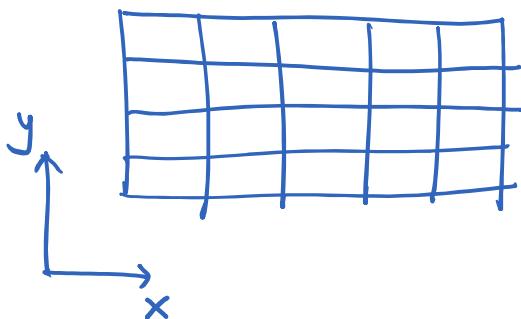
$$\text{filling fraction: } v = \frac{N}{N_L(B)}$$

The reason why Hall conductance is quantized
is still not clear!

* Quantization of Hall conductance

- chiral edge states

Let's consider a *Lattice discretization* of the Hamiltonian for 2D electron gas in a strong magnetic field.



Square Lattice:

$$\hat{a}_1 = a \hat{x}$$

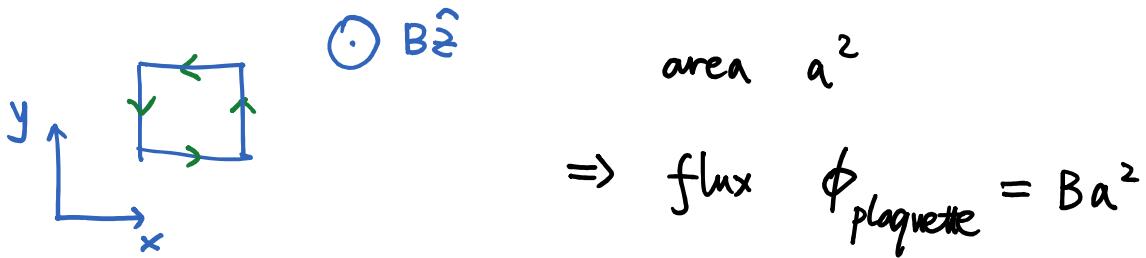
$$\hat{a}_2 = a \hat{y}$$

$$H = \sum_{\langle \vec{r}, \vec{r}' \rangle} t_{\vec{r}, \vec{r}'} (c_{\vec{r}}^+ c_{\vec{r}'} + c_{\vec{r}'}^+ c_{\vec{r}}) + \sum_{\vec{r}} E_0 c_{\vec{r}}^+ c_{\vec{r}}$$

We look for a "smooth" continuum limit such that $a \rightarrow 0$ recovers the Landau levels.

This will also allow us to study boundary effects, which are difficult to address in the continuum.

Consider an elementary plaquette:



Aharanov - Bohm effect:

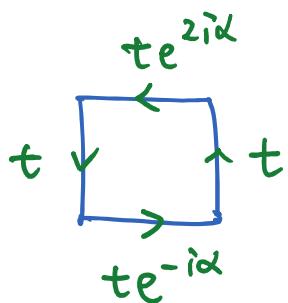
Electron accumulates a phase $e^{i\alpha}$ after moving along the closed path.

$$\alpha = \frac{(-e)}{\hbar} \int_{\text{closed path}} \vec{A} \cdot d\vec{l} = \frac{(-e) \phi_{\text{plaquette}}}{\hbar}$$

$$= -2\pi \frac{\phi_{\text{plaquette}}}{\phi_0} \quad \leftarrow \phi_0 = h/e$$

Pierls substitution:

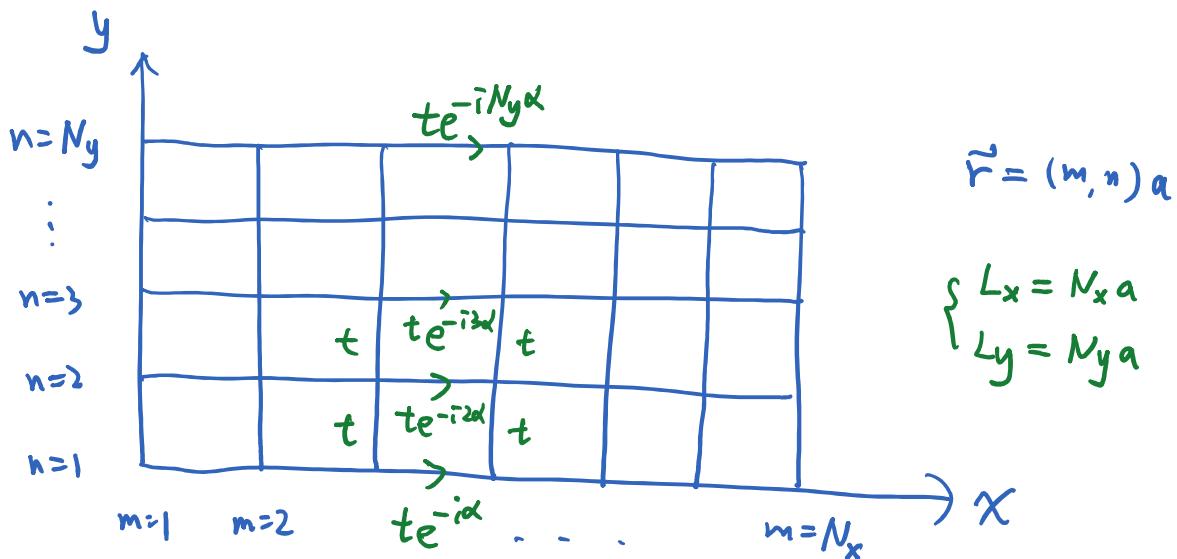
$$t \rightarrow t e^{i \frac{(-e)}{\hbar} \int_{\vec{r}}^{\vec{r}'} \vec{A} \cdot d\vec{l}}$$



Landau gauge

$$\vec{A} = (-B_y, 0, 0)$$

$$H = \sum_{\langle \vec{r}, \vec{r}' \rangle} t_{\vec{r}, \vec{r}'} (C_{\vec{r}}^+ C_{\vec{r}'} + C_{\vec{r}'}^+ C_{\vec{r}}) + \sum_{\vec{r}} E_0 C_{\vec{r}}^+ C_{\vec{r}}$$



Natural choice of boundary conditions:

Periodic boundary condition in x -direction,
Open boundary condition in y -direction.



c.f. $[\hat{H}, \hat{P}_x] = 0$ in the continuum
(Landau gauge)

Diagonalize H :

$$C_{\vec{r}=(m,n)} = \frac{1}{\sqrt{N_x}} \sum_{k_x} C_{k_x, n} e^{ik_x m a} \quad (\text{Fourier transformation in } x\text{-direction})$$

$$k_x = 0, \pm \frac{2\pi}{N_x a}, \dots, \frac{\pi}{a}$$

$$\Rightarrow H = \sum_{k_x} \sum_{n, n'=1}^{N_y} C_{k_x, n}^+ \underbrace{[f_P(k_x)]}_{n, n'} C_{k_x, n'}$$

$$[f_P(k_x)]_{n, n'} = t(\delta_{n', n+1} + \delta_{n', n-1}) + \delta_{n, n'} [2t \cos(k_x a - n\alpha) + E_0]$$

example:

$$\sum_{m=1}^{N_x} (e^{-in\alpha} C_{m, n}^+ C_{m+1, n} + e^{in\alpha} C_{m+1, n}^+ C_{m, n})$$

$$= \frac{1}{N_x} \sum_{m=1}^{N_x} \sum_{k_x, k_x'} \left[e^{-in\alpha} C_{k_x, n}^+ C_{k_x', n} e^{ik_x'(m+1)a - ik_x m a} + e^{in\alpha} C_{k_x, n}^+ C_{k_x', n} e^{ik_x'ma - ik_x'(m+1)a} \right]$$

$$\frac{1}{N_x} \sum_{m=1}^{N_x} e^{i(k_x' - k_x)m a} = \delta_{k_x, k_x'}$$

$$= \sum_{k_x} \left(e^{-in\alpha} C_{k_x, n}^+ C_{k_x, n} e^{ik_x a} + e^{in\alpha} C_{k_x, n}^+ C_{k_x, n} e^{-ik_x a} \right)$$

$$= \sum_{k_x} 2 \cos(k_x a - n\alpha) C_{k_x, n}^+ C_{k_x, n}$$

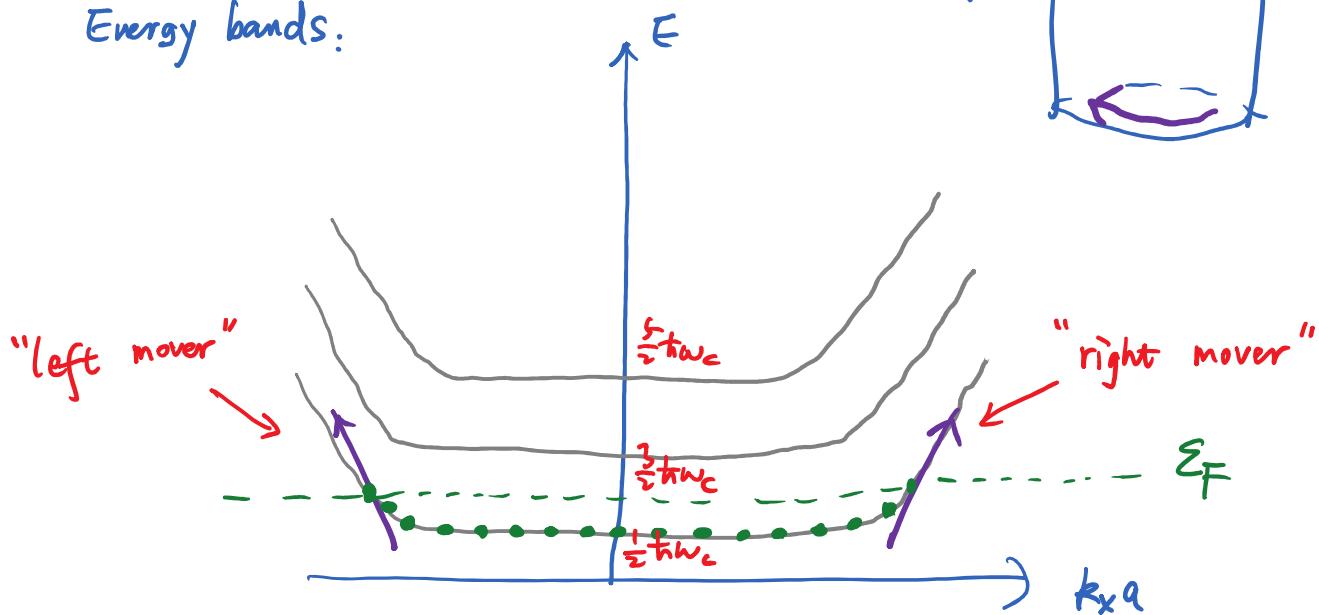
Diagonalizing $N_x \times N_x$ matrix $\tilde{H}(k_x)$ gives single-particle energies: (For each k_x , diagonalizing a 1D Hamiltonian)

$$H = \sum_{k_x} H(k_x)$$

$$\tilde{H}(k_x) = \begin{pmatrix} 2tc_1 & t & 0 & & \\ t & 2tc_2 & t & & \\ 0 & t & 2tc_3 & t & \\ & & t & \ddots & \ddots \end{pmatrix}_{N_x \times N_x} + E_0 \mathbb{1}_{N_x \times N_x}$$

$$C_n = \cos(k_x a - n\alpha)$$

Energy bands:



- Important features:
- 1) Landau levels
 - 2) Chiral edge states at boundaries
(crossing the Fermi energy)
ID conducting channel in x-direction

1D Chiral edge states:

$$H_R \simeq \sum_{k_x} \varepsilon_{k_x, R} \gamma_{k_x, R}^+ \gamma_{k_x, R}^- , \quad \varepsilon_{k_x, R} \simeq v(k_x - k_F)$$

$$H_L \simeq \sum_{k_x} \varepsilon_{k_x, L} \gamma_{k_x, L}^+ \gamma_{k_x, L}^- , \quad \varepsilon_{k_x, L} \simeq v(-k_x - k_F)$$

Remarks:

- 1) chiral edge states **exponentially localized** at two boundaries of the cylinder \Rightarrow This can be checked from the corresponding eigenvectors of $f(k_x)$.
- 2) Translation symmetry not important (disorder, impurities can be present) \leftarrow needs extra work to prove.

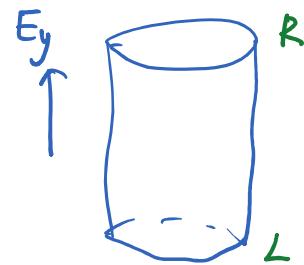
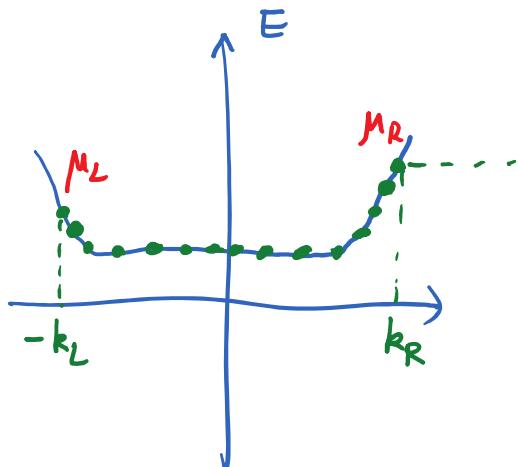
IQH: First example of "topological insulators"

- 3) 1D chiral edge states cannot appear in pure 1D models with local terms on a lattice.

(Nielsen-Ninomiya theorem)

They can only appear as edge states of $D > 1$ topological insulators.

Quantization of Hall conductance:



$$\mu_R - \mu_L = -eV_y$$

(chemical potential imbalanced due to external E_y)

current: $I_x = (-e) \frac{1}{L_x} \sum_{\text{occupied } k_x} V(k_x)$

$$= (-e) \int_{-k_L}^{k_R} \frac{dk_x}{2\pi} \underbrace{V(k_x)}_{\frac{1}{\hbar} \frac{dE(k_x)}{dk_x}}$$

$$= \frac{(-e)}{2\pi\hbar} \int_{\mu_L}^{\mu_R} dE$$

$$= \frac{(-e)}{\hbar} (\mu_R - \mu_L)$$

$$= \frac{e^2}{h} V_y$$

$$\Rightarrow \rho_{xy} = \frac{V_y}{I_x} = \frac{h}{e^2} \quad \text{singly occupied Landau level}$$

(n occupied Landau levels $\Rightarrow n$ pairs of chiral edge states

$$\Rightarrow \rho_{xy} = \frac{1}{n} \frac{h}{e^2}$$