

§5. Quantum Hall effect

* Hall conductance as a bulk topological invariant

- Linear response theory & Kubo formula

$$H = H_0 + H'$$

↑ ↑
 electrons weak external electric field

$$H' = e \vec{E} \cdot \vec{x} \quad \vec{x} = \sum_{j=1}^N \vec{x}_j \quad (N: \# \text{ of electrons})$$

Ground state of H_0 : $|0\rangle$

$$H_0 |0\rangle = \varepsilon_0 |0\rangle$$

Many-electron GS, NOT the usual "vacuum"!

Assume GS unique!

→ This is ok for IQH,
but more subtle for FQH.

Excited states of H_0 :

$$H_0 |n\rangle = \varepsilon_n |n\rangle, \quad n > 0$$

First-order non-degenerate perturbation theory:

$$|\psi\rangle = |0\rangle + \sum_{n>0} \frac{\langle n|H'|0\rangle}{\varepsilon_0 - \varepsilon_n} |n\rangle$$

Electric current:

$$\vec{J} = \frac{-e}{mS} \langle \psi | \vec{\pi} | \psi \rangle$$

area

mechanical momentum,
NOT canonical momentum!

$$\vec{\pi} = m \dot{\vec{x}}$$

$$\dot{\vec{x}} = \frac{i}{\hbar} [H, \vec{x}] = \frac{i}{\hbar} [H_0, \vec{x}]$$

Heisenberg E.O.M.

$$\left. \begin{aligned} & \Rightarrow [\vec{x}, H_0] = \frac{i\hbar}{m} \vec{\pi} \\ & [H', \vec{x}] = 0 \end{aligned} \right\}$$

$$\Rightarrow \vec{J} = -\frac{e}{mS} \left(\langle 0 | + \sum_{n>0} \frac{\langle 0 | H' | n' \rangle}{\varepsilon_0 - \varepsilon_{n'}} \langle n' | \right) \vec{\pi}$$

$$\times \left(|0\rangle + \sum_{n>0} \frac{\langle n | H' | 0 \rangle}{\varepsilon_0 - \varepsilon_n} |n\rangle \right)$$

background current

$$\vec{J}_0 = -\frac{e}{mS} \langle 0 | \vec{\pi} | 0 \rangle = 0$$

(no current without \vec{E})

$$\tilde{J} = \frac{-e}{ms} \sum_{n>0} \frac{\langle o | H' | n \rangle \langle n | \vec{\pi}(0) + \langle n | H' | 0 \rangle \langle o | \vec{\pi}(n) \rangle}{\varepsilon_0 - \varepsilon_n}$$

$+ O(E^2)$

dropped as we consider linear response in \vec{E} !

$$H' = e \vec{E} \cdot \vec{x}$$

$$\simeq \frac{-e}{ms} \sum_{n>0} \frac{e \vec{E} \cdot \langle o | \vec{x} | n \rangle \langle n | \vec{\pi} | 0 \rangle + e \vec{E} \cdot \langle n | \vec{x} | 0 \rangle \langle o | \vec{\pi} | n \rangle}{\varepsilon_0 - \varepsilon_n}$$

$$\langle o | \vec{x} | n \rangle = \frac{\langle o | H_0 \vec{x} - \vec{x} H_0 | n \rangle}{\varepsilon_0 - \varepsilon_n}$$

$\begin{aligned} \langle o | H_0 &= \langle o | \varepsilon_0 \\ H_0 | n \rangle &= \varepsilon_n | n \rangle \end{aligned}$

$$= \frac{\langle o | [H_0, \vec{x}] | n \rangle}{\varepsilon_0 - \varepsilon_n}$$

$[H_0, \vec{x}] = -\frac{i\hbar}{m} \vec{\pi}$

$$= -\frac{i\hbar}{m} \frac{\langle o | \vec{\pi} | n \rangle}{\varepsilon_0 - \varepsilon_n}$$

$$\Rightarrow \tilde{J} = \frac{ie^2 \hbar}{m^2 s} \sum_{n>0} \frac{\vec{E} \cdot \langle o | \vec{\pi} | n \rangle \langle n | \vec{\pi} | 0 \rangle - \vec{E} \cdot \langle n | \vec{\pi} | 0 \rangle \langle o | \vec{\pi} | n \rangle}{(\varepsilon_0 - \varepsilon_n)^2}$$

$$J_\mu = \frac{ie^2 \hbar}{m^2 s} \sum_{n>0} \frac{\langle n | \pi^\mu | 0 \rangle \langle o | \pi^\nu | n \rangle - \langle n | \pi^\nu | 0 \rangle \langle o | \pi^\mu | n \rangle}{(\varepsilon_0 - \varepsilon_n)^2} E_\nu$$

$$(\mu, \nu = x, y, z)$$

Compare with $J_\mu = \sum_\nu \sigma_{\mu\nu} E_\nu$:

$$\sigma_{\mu\nu} = \frac{ie^2\hbar}{m^2 S} \sum_{n>0} \frac{\langle n | \pi_\mu | 0 \rangle \langle 0 | \pi_\nu | n \rangle - \langle n | \pi_\nu | 0 \rangle \langle 0 | \pi_\mu | n \rangle}{(\varepsilon_0 - \varepsilon_n)^2}$$

$$= \frac{e^2 \hbar}{i S} \sum_{n>0} \frac{\langle 0 | v_\mu | n \rangle \langle n | v_\nu | 0 \rangle - \langle 0 | v_\nu | n \rangle \langle n | v_\mu | 0 \rangle}{(\varepsilon_0 - \varepsilon_n)^2}$$

velocity operator: $v_\mu = \frac{\pi_\mu}{m}$

This is the Kubo formula for DC conductivity!
(not restricted to quantum Hall systems)

- Quantized Hall conductance

Landau gauge:

$$\vec{A} = (0, Bx, 0)$$

$$H = \sum_{j=1}^N \left[\frac{1}{2m_j} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} \right)^2 + \frac{1}{2m_j} \left(\frac{\hbar}{i} \frac{\partial}{\partial y_j} - eBx_j \right)^2 \right]$$

\downarrow
electrons may have different effective masses

$$+ \sum_{j=1}^N U(x_j, y_j) + \sum_{i < j} V(|\vec{r}_i - \vec{r}_j|)$$

\uparrow
impurity potential

\uparrow
electron-electron interactions

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(5)

Velocity operators :

$$v_x = \sum_{j=1}^N \frac{1}{m_j} \cdot \underbrace{\frac{m_j}{i\hbar} [x_j, H_0]}_{\text{II}} = \sum_{j=1}^N \frac{1}{m_j} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} \right)$$

$\pi_{j,x}$

$$v_y = \sum_{j=1}^N \frac{1}{m_j} \cdot \underbrace{\frac{m_j}{i\hbar} [y_j, H_0]}_{\text{II}} = \sum_{j=1}^N \frac{1}{m_j} \left(\frac{\hbar}{i} \frac{\partial}{\partial y_j} - eB x_j \right)$$

$\pi_{j,y}$

Hall conductance :

$$\sigma_{xy} = \frac{e^2 \hbar}{iS} \sum_{n>0} \frac{\langle 0 | v_x | n \rangle \langle n | v_y | 0 \rangle - \langle 0 | v_y | n \rangle \langle n | v_x | 0 \rangle}{(\varepsilon_0 - \varepsilon_n)^2}$$

(Not clear at this stage why σ_{xy} is quantized ...)

Define unitary transformation :

$$U = e^{-i\alpha \sum_{j=1}^N x_j} e^{-i\beta \sum_{j=1}^N y_j}$$

Note that $e^{-i\alpha x} p e^{i\alpha x} = p + \alpha \hbar$,so U can shift momenta!

Hamiltonian after unitary transformation:

$$\begin{aligned}\tilde{H}(\alpha, \beta) &= U H U^{-1} \\ &= \sum_{j=1}^N \left[\frac{1}{2m_j} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} + \alpha \hbar \right)^2 + \frac{1}{2m_j} \left(\frac{\hbar}{i} \frac{\partial}{\partial y_j} + \beta \hbar - eBx_j \right)^2 \right] \\ &\quad + \sum_{j=1}^N V(x_j, y_j) + \sum_{i < j} V(|\vec{r}_i - \vec{r}_j|)\end{aligned}$$

Velocity operators in the new basis:

$$\begin{aligned}\tilde{v}_x &= U v_x U^{-1} \\ &= \sum_{j=1}^N \frac{1}{m_j} U \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} \right) U^{-1} \\ &= \sum_{j=1}^N \frac{1}{m_j} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} + \alpha \hbar \right)\end{aligned}$$

$$\begin{aligned}\tilde{v}_y &= U v_y U^{-1} \\ &= \sum_{j=1}^N \frac{1}{m_j} \left(\frac{\hbar}{i} \frac{\partial}{\partial y_j} + \beta \hbar - eBx_j \right)\end{aligned}$$

It's easy to observe that $\frac{\partial \tilde{H}(\alpha, \beta)}{\partial \alpha} = \hbar v_x$

$$\frac{\partial \tilde{H}(\alpha, \beta)}{\partial \beta} = \hbar v_y$$

σ_{xy} in the new basis:

$$\begin{aligned}\sigma_{xy} &= \frac{e^2}{i\hbar} \sum_{n>0} \frac{\langle 0 | \tilde{V}_x | n \rangle \langle n | \tilde{V}_y | 0 \rangle - \langle 0 | \tilde{V}_y | n \rangle \langle n | \tilde{V}_x | 0 \rangle}{(\varepsilon_0 - \varepsilon_n)^2} \\ &= \frac{e^2}{i\hbar k} \sum_{n>0} \frac{\langle 0 | \frac{\partial \tilde{H}}{\partial \alpha} | n \rangle \langle n | \frac{\partial \tilde{H}}{\partial \beta} | 0 \rangle - \langle 0 | \frac{\partial \tilde{H}}{\partial \beta} | n \rangle \langle n | \frac{\partial \tilde{H}}{\partial \alpha} | 0 \rangle}{(\varepsilon_0 - \varepsilon_n)^2}\end{aligned}$$

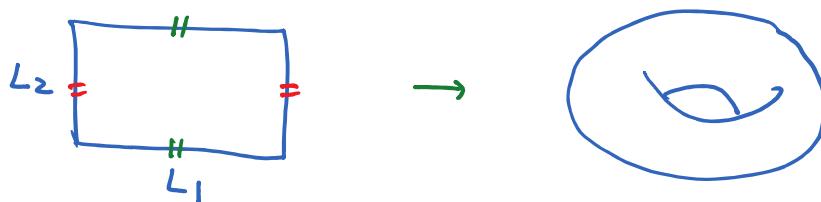
Remark: $|0\rangle$ and $|n\rangle$ now depend on α and β !

Better notation would be $|\phi_0(\alpha, \beta)\rangle$ for $|0\rangle$.

$$\begin{aligned}&\langle 0 | \frac{\partial \tilde{H}}{\partial \alpha} | n \rangle \langle n | \frac{\partial \tilde{H}}{\partial \beta} | 0 \rangle \\ &= \left(\frac{\partial}{\partial \alpha} \underbrace{\langle 0 | \tilde{H} | n \rangle}_{\varepsilon_0 \langle 0 | n \rangle = 0} - \underbrace{\langle \frac{\partial}{\partial \alpha} 0 | \tilde{H} | n \rangle}_{\varepsilon_n \langle \frac{\partial}{\partial \alpha} 0 | n \rangle} - \underbrace{\langle 0 | \tilde{H} | \frac{\partial}{\partial \alpha} n \rangle}_{\varepsilon_0 \langle 0 | \frac{\partial}{\partial \alpha} n \rangle = -\varepsilon_0 \langle \frac{\partial}{\partial \alpha} 0 | n \rangle} \right) \\ &\quad \times \left(\frac{\partial}{\partial \beta} \underbrace{\langle n | \tilde{H} | 0 \rangle}_{0} - \underbrace{\langle \frac{\partial}{\partial \beta} n | \tilde{H} | 0 \rangle}_{\varepsilon_0 \langle \frac{\partial}{\partial \beta} n | 0 \rangle = -\varepsilon_0 \langle n | \frac{\partial}{\partial \beta} 0 \rangle} - \underbrace{\langle n | \tilde{H} | \frac{\partial}{\partial \beta} 0 \rangle}_{\varepsilon_n \langle n | \frac{\partial}{\partial \beta} 0 \rangle} \right) \\ &= (\varepsilon_0 - \varepsilon_n)^2 \langle \frac{\partial}{\partial \alpha} 0 | n \rangle \langle n | \frac{\partial}{\partial \beta} 0 \rangle\end{aligned}$$

$$\begin{aligned}\sigma_{xy} &= \frac{e^2}{i\hbar k} \sum_{n>0} \frac{\langle 0 | \hat{\frac{\partial H}{\partial \alpha}} | n \rangle \langle n | \hat{\frac{\partial H}{\partial \beta}} | 0 \rangle - \langle 0 | \hat{\frac{\partial H}{\partial \beta}} | n \rangle \langle n | \hat{\frac{\partial H}{\partial \alpha}} | 0 \rangle}{(\varepsilon_0 - \varepsilon_n)^2} \\ &= \frac{e^2}{i\hbar k} \sum_{n>0} \left[\underbrace{\langle \frac{\partial}{\partial \alpha} 0 | n \rangle \langle n | \frac{\partial}{\partial \beta} 0 \rangle}_{\sum_{n>0} |n\rangle \langle n| = 1 - |0\rangle \langle 0|} - \underbrace{\langle \frac{\partial}{\partial \beta} 0 | n \rangle \langle n | \frac{\partial}{\partial \alpha} 0 \rangle}_{\sum_{n>0} |n\rangle \langle n| = 1 - |0\rangle \langle 0|} \right] \\ &= \frac{e^2}{i\hbar k} \left(\langle \frac{\partial}{\partial \alpha} 0 | \frac{\partial}{\partial \beta} 0 \rangle - \langle \frac{\partial}{\partial \beta} 0 | \frac{\partial}{\partial \alpha} 0 \rangle \right)\end{aligned}$$

Define $\theta = \alpha L_1, \varphi = \beta L_2 :$



"twist boundary condition",

parametrized by two angles $\theta \in [0, 2\pi], \varphi \in [0, 2\pi]$.

$$\sigma_{xy} = \frac{e^2}{i\hbar} \left(\langle \frac{\partial}{\partial \theta} 0 | \frac{\partial}{\partial \varphi} 0 \rangle - \langle \frac{\partial}{\partial \varphi} 0 | \frac{\partial}{\partial \theta} 0 \rangle \right)$$

We expect that σ_{xy} should be **insensitive** to boundary conditions for L_1, L_2 large and $N \rightarrow \infty$.

Thus, we could average σ_{xy} over all possible twisted boundary conditions $\theta \in [0, 2\pi)$, $\varphi \in [0, 2\pi)$:

$$\bar{\sigma}_{xy} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\theta d\varphi \frac{e^2}{i\hbar} \left(\langle \frac{\partial}{\partial \theta} \sigma | \frac{\partial}{\partial \varphi} \sigma \rangle - \langle \frac{\partial}{\partial \varphi} \sigma | \frac{\partial}{\partial \theta} \sigma \rangle \right)$$

$$= \frac{e^2}{h} \int_0^{2\pi} \int_0^{2\pi} d\theta d\varphi \frac{1}{2\pi i} \left(\langle \frac{\partial}{\partial \theta} \sigma | \frac{\partial}{\partial \varphi} \sigma \rangle - \langle \frac{\partial}{\partial \varphi} \sigma | \frac{\partial}{\partial \theta} \sigma \rangle \right)$$

↑

topological invariant!

(The first Chern number \Rightarrow integers)

This provides a non-perturbative proof that

σ_{xy} must be quantized, as it is related to a topological invariant.

Ref.: Q. Niu, D.J. Thouless, & Y.S. Wu,
Phy. Rev. B 31, 3372 (1985).