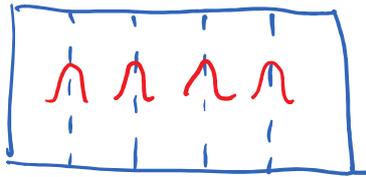


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①

§5. Quantum Hall effect

* FQH: paragon picture

$$\frac{\hbar}{eB} \frac{2\pi}{L_x} = \frac{2\pi l^2}{L_x}$$

magnetic length:

$$l = \sqrt{\frac{\hbar}{eB}}$$

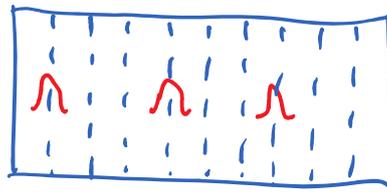
IQH: $\nu = \text{integers}$ (fully filled Landau levels)

Q: What happens if Landau levels are partially filled? (Experiment: Larger B-field)

Macroscopic degeneracy if interactions are not considered.

However, electron interactions will make a choice!

Naive expectation:



Wigner crystal (electrons try to be far from each other to minimize repulsion)

But this doesn't explain FQH plateaux ...

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②

— "Parton" picture

Let us consider $\nu = 1/3$ case:



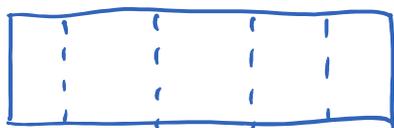
electron

parton

$$C^+(\vec{r}) \leftrightarrow d_1^+(\vec{r}) d_2^+(\vec{r}) d_3^+(\vec{r})$$

d_a^+ carry electronic charge $-\frac{e}{3}$!
 \downarrow
 $a=1,2,3$

Landau levels for partons:



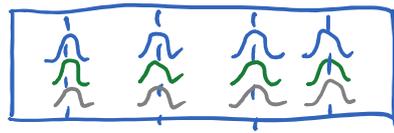
$$\underbrace{\frac{\hbar}{3eB} \frac{2\pi}{L_x}} = 3 \frac{\hbar}{eB} \frac{2\pi}{L_x}$$

distance between two single-particle states **three** times larger!

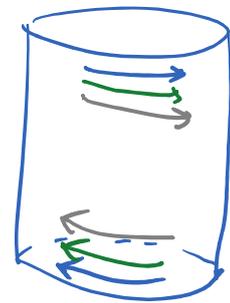
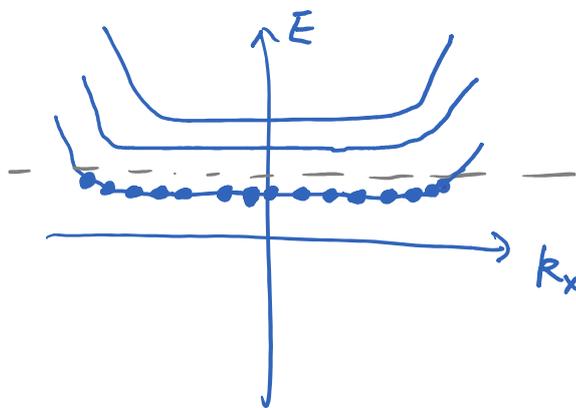
$\nu = 1/3$ for electrons = fully filled LLL of partons

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③



Each single-particle level can accommodate **three partons** with different colors ($a=1,2,3$).



$$\Rightarrow \sigma_{xy} = 3 \cdot \frac{\left(\frac{e}{3}\right)^2}{h} = \frac{1}{3} \frac{e^2}{h}$$

Remark: Many other FQH plateaux can be explained with a similar parton picture.

But it requires careful justification with microscopic calculations!

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④

Laughlin's wave function from the parton construction:

$$|4\rangle = P \prod_{a=1,2,3} \prod_m \psi_{m,a}^+ |0\rangle$$

↑ ↘

enforce $C^+(\vec{r}) = d_1^+(\vec{r}) d_2^+(\vec{r}) d_3^+(\vec{r})$
v=1 for partons

$$\psi_{m,a}^+ = \int d\vec{r} \underbrace{\phi_m(\vec{r})}_{\downarrow} d_a^+(\vec{r})$$

LLL single-particle wave function:

$$\phi_m(\vec{r}) \propto z^m e^{-|z|^2/2l_p^2} \quad (\text{disk geometry})$$

c.f. Ex. 1 in Problem Set 5

Note that $l_p^2 = \frac{\hbar}{\frac{e}{3}B} = 3l^2$

l_p : magnetic length
for partons

l : magnetic length
for electrons

$$|4\rangle \propto P \prod_{a=1,2,3} \left[\int d\vec{r}_{1,a} \cdots d\vec{r}_{N,a} \det \begin{pmatrix} 1 & \cdots & 1 \\ z_{1,a} & z_{2,a} & \cdots & z_{N,a} \\ \vdots & \vdots & \ddots & \vdots \\ z_{1,a}^{N-1} & z_{2,a}^{N-1} & \cdots & z_{N,a}^{N-1} \end{pmatrix} \right]$$

||
 $\prod_{i < j} (z_{i,a} - z_{j,a})$

$$\times e^{\sum_j \frac{|z_{j,a}|^2}{2l_p^2}} d_a^+(\vec{r}_{1,a}) \cdots d_a^+(\vec{r}_{N,a}) \Big] |0\rangle$$

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⑤

P requires that $\vec{r}_{j,\alpha=1} = \vec{r}_{j,\alpha=2} = \vec{r}_{j,\alpha=3}$.

partons with different colors
form an electron!

$$|4\rangle \propto P \int d\vec{r}_1 \dots d\vec{r}_N \left[\prod_{i<j} (z_i - z_j) e^{-\sum_j \frac{|z_j|^2}{2\ell^2}} \right]^3$$

$$\times \underbrace{d_1^+(\vec{r}_1) d_2^+(\vec{r}_1) d_3^+(\vec{r}_1)}_{C^+(\vec{r}_1)} \dots \underbrace{d_1^+(\vec{r}_N) d_2^+(\vec{r}_N) d_3^+(\vec{r}_N)}_{C^+(\vec{r}_N)} |0\rangle$$

$$= \int d\vec{r}_1 \dots d\vec{r}_N \prod_{i<j} (z_i - z_j)^3 e^{-\sum_j \frac{|z_j|^2}{2\ell^2}} C^+(\vec{r}_1) \dots C^+(\vec{r}_N) |0\rangle$$

↓
Laughlin's wave function for $\nu = 1/3$!

Numerics show that Laughlin's wave function

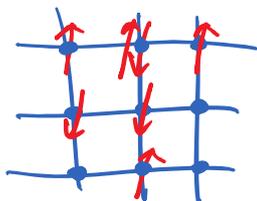
is indeed a very good **variational ansatz**

for the microscopic model ($\nu = 1/3$ with

electron interactions projected to the LLL)!

§6. Quantum magnetism

*) Super-exchange interaction and Heisenberg models



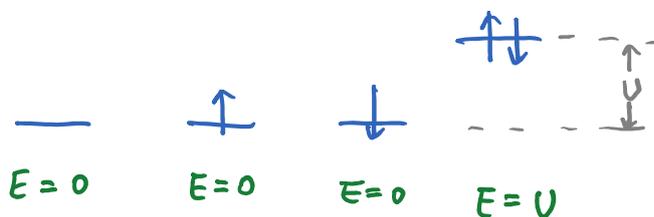
Hubbard model:

$$H = -t \sum_{\langle ij \rangle} \sum_{\sigma=\uparrow,\downarrow} (C_{i\sigma}^{\dagger} C_{j\sigma} + C_{j\sigma}^{\dagger} C_{i\sigma}) + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

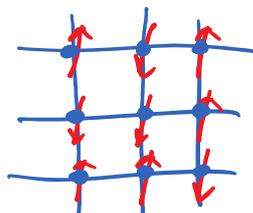
$n_{i\sigma} = C_{i\sigma}^{\dagger} C_{i\sigma}$

half-filling: $N_e = N$

$$\epsilon_{\mathbf{k}} = -2t(\cos k_x + \cos k_y)$$

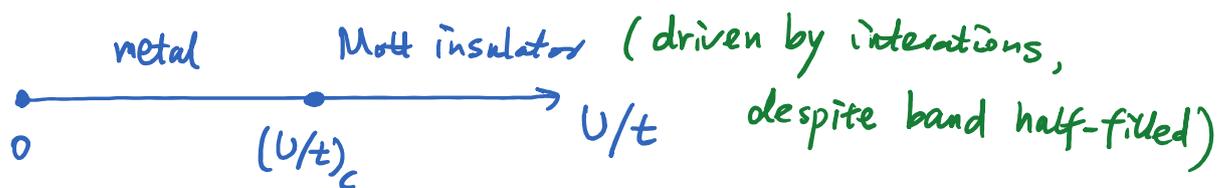
metal for $U=0!$ $U > 0?$ single site: $H_U = U n_{\uparrow} n_{\downarrow}$  $U \rightarrow \infty$:
half-filled

single occupancy

 2^N configurations

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(7)



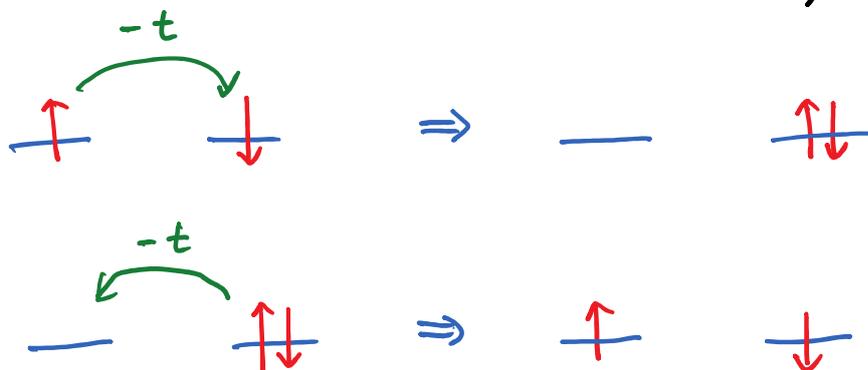
$U \gg t$: second-order perturbation theory (in t/U)

$$H_{\text{eff}}^{(2)} = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \quad s = 1/2$$

singly-occupied electron

$$J = \frac{4t^2}{U} > 0 \quad (\text{AFM interaction})$$

virtual process in second-order perturbation theory:



(gain energy $\sim t^2/U$ when antiparallel)



(does not gain energy from hopping when parallel)

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⑧

More careful derivation of the **super-exchange** interaction:

Two-site low-energy subspace:

$$|\uparrow, \uparrow\rangle, |\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle, |\downarrow, \downarrow\rangle$$

$$1 = P_0 + P_1$$

P_0 : project onto the singly occupied subspace

P_1 : orthogonal complement

$$\langle a | H_{\text{eff}}^{(2)} | b \rangle = \langle a | H_t P_1 \frac{1}{E^{(0)} - H_0} P_1 H_t | b \rangle$$

$\begin{array}{c} \downarrow \quad \swarrow \\ \text{low-energy} \\ \text{subspace} \end{array}$

$$-t(C_{1\sigma}^\dagger C_{2\sigma} + C_{2\sigma}^\dagger C_{1\sigma}) \quad U(n_{1\uparrow}n_{1\downarrow} + n_{2\uparrow}n_{2\downarrow})$$

$$\langle \uparrow \uparrow | H_{\text{eff}}^{(2)} | \uparrow \uparrow \rangle = \langle \uparrow \uparrow | -t(C_{1\sigma}^\dagger C_{2\sigma} + C_{2\sigma}^\dagger C_{1\sigma}) P_1$$

$$\times \frac{-1}{H_0} P_1 (-t) \underbrace{(C_{1\sigma}^\dagger C_{2\sigma} + C_{2\sigma}^\dagger C_{1\sigma}) C_{1\uparrow}^\dagger C_{2\uparrow}^\dagger | 0 \rangle}_{=0}$$

$$= 0$$

$$\text{e.g. } C_{1\uparrow}^\dagger C_{2\uparrow} C_{1\uparrow}^\dagger C_{2\uparrow}^\dagger | 0 \rangle = 0$$

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⑨

$$\begin{aligned}
 \langle \uparrow\downarrow | H_{\text{eff}}^{(2)} | \uparrow\downarrow \rangle &= \underbrace{\langle 0 | C_{2\downarrow} C_{1\uparrow} (-t) (C_{1\sigma}^\dagger C_{2\sigma} + C_{2\sigma}^\dagger C_{1\sigma}) P_1}_{(-t) (\langle 0 | C_{2\downarrow} C_{2\uparrow} + \langle 0 | C_{1\downarrow} C_{1\uparrow})} \\
 &\quad \times \frac{-1}{H_0} P_1 (-t) \underbrace{(C_{1\sigma}^\dagger C_{2\sigma} + C_{2\sigma}^\dagger C_{1\sigma}) C_{1\uparrow}^\dagger C_{2\downarrow}^\dagger | 0 \rangle}_{C_{1\uparrow}^\dagger C_{1\downarrow}^\dagger | 0 \rangle + C_{2\uparrow}^\dagger C_{2\downarrow}^\dagger | 0 \rangle} \\
 &= - \frac{2t^2}{U}
 \end{aligned}$$

$$\begin{aligned}
 \langle \uparrow\downarrow | H_{\text{eff}}^{(2)} | \downarrow\uparrow \rangle &= \underbrace{\langle 0 | C_{2\downarrow} C_{1\uparrow} (-t) (C_{1\sigma}^\dagger C_{2\sigma} + C_{2\sigma}^\dagger C_{1\sigma}) P_1}_{(-t) (\langle 0 | C_{2\downarrow} C_{2\uparrow} + \langle 0 | C_{1\downarrow} C_{1\uparrow})} \\
 &\quad \times \frac{-1}{H_0} P_1 (-t) \underbrace{(C_{1\sigma}^\dagger C_{2\sigma} + C_{2\sigma}^\dagger C_{1\sigma}) C_{1\downarrow}^\dagger C_{2\uparrow}^\dagger | 0 \rangle}_{-C_{1\uparrow}^\dagger C_{1\downarrow}^\dagger | 0 \rangle - C_{2\uparrow}^\dagger C_{2\downarrow}^\dagger | 0 \rangle} \\
 &= + \frac{2t^2}{U}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow H_{\text{eff}}^{(2)} &= \frac{2t^2}{U} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{matrix} \\
 &= \frac{t^2}{U} (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z - \mathbb{1}_{2 \times 2} \otimes \mathbb{1}_{2 \times 2}) \\
 &= \frac{4t^2}{U} (\vec{S}_1 \cdot \vec{S}_2 - \frac{1}{4}) \quad \checkmark
 \end{aligned}$$