Posted: 24.06.2020
Tutorial: 25.06.2020
Due: 08.07.2020

Problem Set 6

1. **Wave packet picture in Landau gauge** (3+5 points)

Consider an electron in two dimensions (x-y plane) with a perpendicular uniform magnetic field corresponding to the vector potential $\vec{A} = -By\hat{x}$ (i.e., Landau gauge).

(a) Verify that the single-particle wave functions in the lowest Landau level (LLL) can be written as

$$\psi_{k_x}(x,y) \propto \frac{1}{\sqrt{L_x}} e^{ik_x x} e^{-(y-y_0)^2/(2l^2)},$$

where $y_0 = l^2 k_x$ with $k_x = 0, \pm\frac{2\pi}{L_x}, \pm\frac{4\pi}{L_x}, \ldots$ and $l = \sqrt{\frac{\hbar}{eB}}$ is the magnetic length.

(b) The wave functions $\psi_{k_x}(x,y)$ are localized along the y direction but extended along the x direction. Since they all have the same energy, one may find proper linear superpositions of them, so that the resulting wave function describes a wave packet (i.e., spatially localized in both direction),

$$\phi(x,y) = \sum_{k_x} a_{k_x} \psi_{k_x}(x,y).$$

Can you find a suitable choice of a_{k_x} such that $\phi(x,y)$ corresponds to a wave packet?

Hint: The calculation simplifies in the limit $L_x \to \infty$.

2. **Landau levels in symmetric gauge** (2+2+2+2+2 points)

In the symmetric gauge, the Hamiltonian for an electron moving in two dimensions (x-y plane) under a perpendicular magnetic field $\vec{B} = B\hat{z}$ is given by

$$H = \frac{1}{2m}(p_x + eA_x)^2 + \frac{1}{2m}(p_y + eA_y)^2$$

with $\vec{A} = (A_x, A_y, 0) = (-\frac{B}{2}y, \frac{B}{2}x, 0)$. For convenience, one may set $m = e = B = \hbar = 1$.

(a) Introduce complex coordinates $z = x - iy$ and $\bar{z} = x + iy$. Show that the Hamiltonian can be rewritten as

$$H = \frac{1}{2}\left(-4\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{4}z\bar{z} - z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right).$$

(b) Introduce bosonic operators

$$a = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right),$$
$$b = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right).$$

Show that these bosonic operators indeed satisfy the standard commutation relations, i.e., $[a, a^\dagger] = [b, b^\dagger] = 1$ (other commutators are zero).

(c) Show that the Hamiltonian is simply written as

$$H = a^\dagger a + \frac{1}{2},$$

and the eigenvalues/eigenvectors of the Landau levels are given by

$$H|n,m\rangle = E_n|n,m\rangle,$$

where $E_n = n + \frac{1}{2}$ and $|n,m\rangle = \frac{(b^\dagger)^{m+n}}{\sqrt{(m+n)!}}\frac{(a^\dagger)^n}{\sqrt{n!}}|0,0\rangle$ $(m = -n, -n+1, \ldots)$.

(d) Show that the single-electron wave functions in the lowest Landau level take the following form in coordinate space:

$$\phi_{0,m}(z,\bar{z}) = \langle \mathbf{r}|0,m\rangle \propto z^m e^{-|z|^2/4}.$$

You are encouraged to derive the normalization constant (but not required to do so).

(e) What is the many-electron wave function when the lowest Landau level is fully occupied (i.e., filling fraction $\nu = 1$)?