SFB 1143

Solid State Theory (SS2020)

Lecture 18: Integer quantum Hall effect

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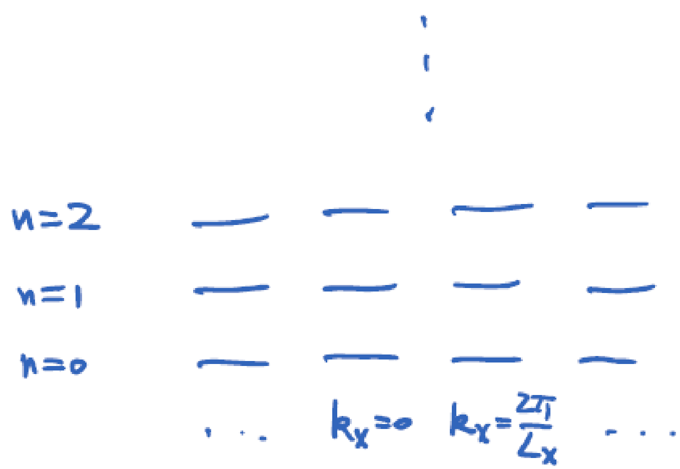
July 1st, 2020

§ 5.2 Integer quantum Hall effect

- Last lecture: Landau levels

$$H = \frac{1}{2m}\left(\hat{p}_x - eB\hat{y}\right)^2 + \frac{1}{2m}\hat{p}_y^2$$

$$E_{n,k_x} = \left(n+\frac{1}{2}\right)\hbar\omega_c \qquad \omega_c = \frac{eB}{m}$$



§ 5.2 Integer quantum Hall effect

- Last lecture: Landau levels

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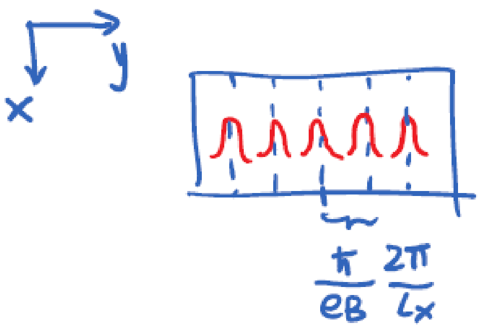

degeneracy:

$$N_L(B) = \frac{L_y}{\frac{\hbar}{eB}\frac{2\pi}{L_x}} = \frac{BL_xL_y}{h/e} = \frac{\phi}{\phi_0}$$

filling fraction: $\nu = \frac{N}{N_L(B)}$

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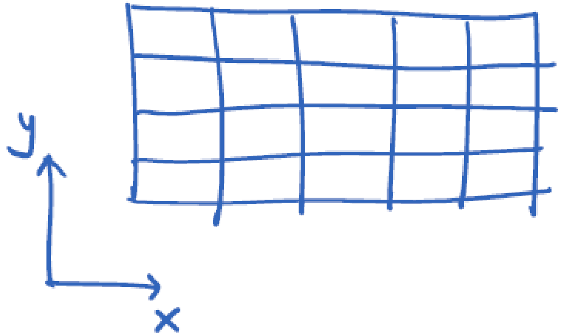
- Quantized Hall conductance (edge state picture)
  - Consider a lattice discretization of the Hamiltonian



square lattice:

$$\hat{a}_1 = a\hat{x}$$
$$\hat{a}_2 = a\hat{y}$$

We look for a "smooth" continuum limit such that $a \to 0$ recovers the Landau levels.

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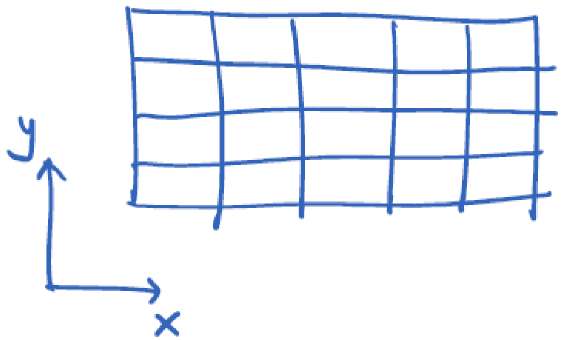
- Quantized Hall conductance (edge state picture):
  - ➢ Consider a lattice discretization of the Hamiltonian



square lattice:

$$\hat{a}_1 = a\hat{x}$$

$$\hat{a}_2 = a\hat{y}$$

$$H = \sum_{\langle \vec{r}, \vec{r}' \rangle} t_{\vec{r}, \vec{r}'} \left(C^\dagger_{\vec{r}} C_{\vec{r}'} + C^\dagger_{\vec{r}'} C_{\vec{r}} \right) + \sum_{\vec{r}} E_0 C^\dagger_{\vec{r}} C_{\vec{r}}$$

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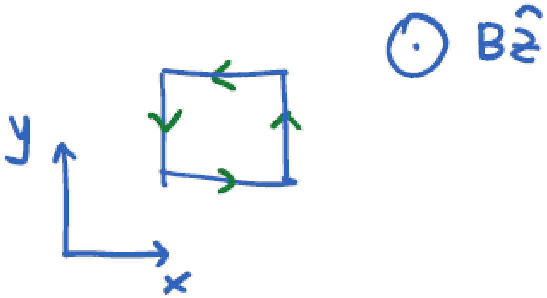
Consider an elementary plaquette:



area a^2

$\Rightarrow$ flux $\phi_{\text{plaquette}} = Ba^2$

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- Aharonov-Bohm effect:

Electron accumulates a phase $e^{i\alpha}$ after moving along the closed path.

$$\alpha = \frac{(-e)}{\hbar} \int_{\square} \vec{A} \cdot d\vec{l} = \frac{(-e)\,\phi_{\text{plaquette}}}{\hbar}$$

$$= -2\pi \, \frac{\phi_{\text{plaquette}}}{\phi_0}$$

$\phi_0 = h/e$

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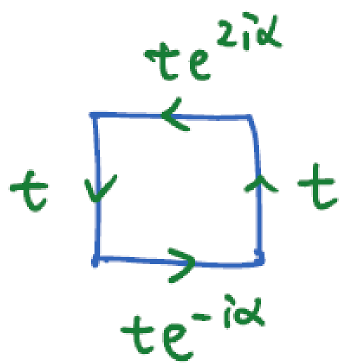
- Aharonov-Bohm effect:

Peierls substitution:

$$t \longrightarrow t e^{i \frac{(-e)}{\hbar} \int_{\vec{r}}^{\vec{r}\,'} \vec{A} \cdot d\vec{\ell}}$$



Landau gauge

$$\vec{A} = (-By,\ 0,\ 0)$$

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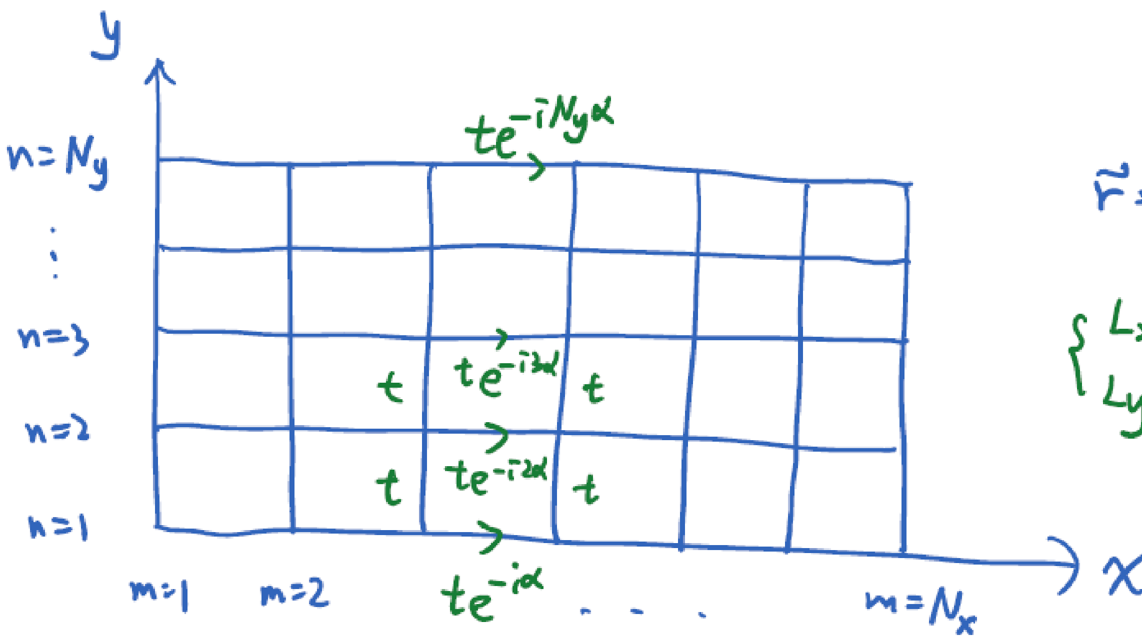
$$H = \sum_{\langle \vec{r}, \vec{r}' \rangle} t_{\vec{r},\vec{r}'} \left(C^{\dagger}_{\vec{r}} C_{\vec{r}'} + C^{\dagger}_{\vec{r}'} C_{\vec{r}} \right) + \sum_{\vec{r}} E_0 C^{\dagger}_{\vec{r}} C_{\vec{r}}$$



$\vec{r} = (m, n)a$

$$\begin{cases} L_x = N_x a \\ L_y = N_y a \end{cases}$$

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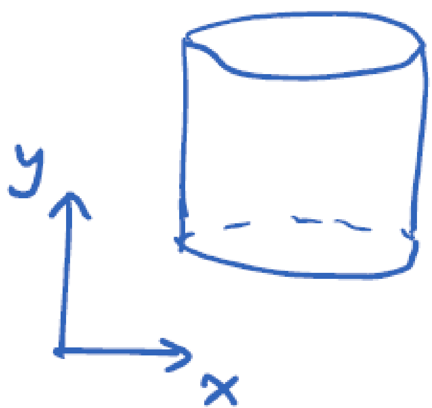

"Cylinder geometry"

Periodic boundary condition in x-direction,

Open boundary condition in y-direction.

c.f. $[\hat{H}, \hat{P}_x] = 0$ in the continuum

(Landau gauge)

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$$H = \sum_{\langle \vec{r}, \vec{r}\,' \rangle} t_{\vec{r}, \vec{r}\,'} \left(C^{+}_{\vec{r}} C_{\vec{r}\,'} + C^{+}_{\vec{r}\,'} C_{\vec{r}} \right) + \sum_{\vec{r}} E_0 C^{+}_{\vec{r}} C_{\vec{r}}$$

Diagonalize H :

$$C_{\vec{r}=(m,n)} = \frac{1}{\sqrt{N_x}} \sum_{k_x} C_{k_x, n} \, e^{i k_x m a}$$

(Fourier transformation in x-direction)

$k_x = 0, \pm \frac{2\pi}{N_x a}, \ldots, \frac{\pi}{a}$

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$$H = \sum_{k_x} \sum_{n,n'=1}^{N_y} C^{+}_{k_x,n} \left[\mathcal{H}(k_x) \right]_{n,n'} C_{k_x,n'}$$

$$\left[\mathcal{H}(k_x) \right]_{n,n'} = t\left(\delta_{n',n+1} + \delta_{n',n-1} \right) + \delta_{n,n'} \left[2t\cos(k_x a - n\alpha) + E_0 \right]$$

example:

$$\sum_{m=1}^{N_x} \left(e^{-in\alpha} C^{+}_{m,n} C_{m+1,n} + e^{in\alpha} C^{+}_{m+1,n} C_{m,n} \right)$$

$$= \sum_{k_x} \Big(e^{-in\alpha} C^{+}_{k_x,n} C_{k_x,n}\, e^{ik_x a}$$

$$+ e^{in\alpha} C^{+}_{k_x,n} C_{k_x,n}\, e^{-ik_x a} \Big)$$

$$= \sum_{k_x} 2\cos(k_x a - n\alpha)\, C^{+}_{k_x,n} C_{k_x,n}$$

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Diagonalizing $N_y \times N_y$ matrix $\mathcal{H}(k_x)$ gives single-particle energies: (For each k_x, diagonalize a 1D Hamiltonian)

$$H = \sum_{k_x} H(k_x)$$

$$\mathcal{H}(k_x) = \begin{pmatrix} 2tc_1 & t & 0 & & \\ t & 2tc_2 & t & & \\ 0 & t & 2tc_3 & t & \\ & & t & \ddots & \\ & & & & \ddots \end{pmatrix}_{N_y \times N_y} + E_0 \mathbb{1}_{N_y \times N_y}$$

$$c_n = \cos(k_x a - n\alpha)$$

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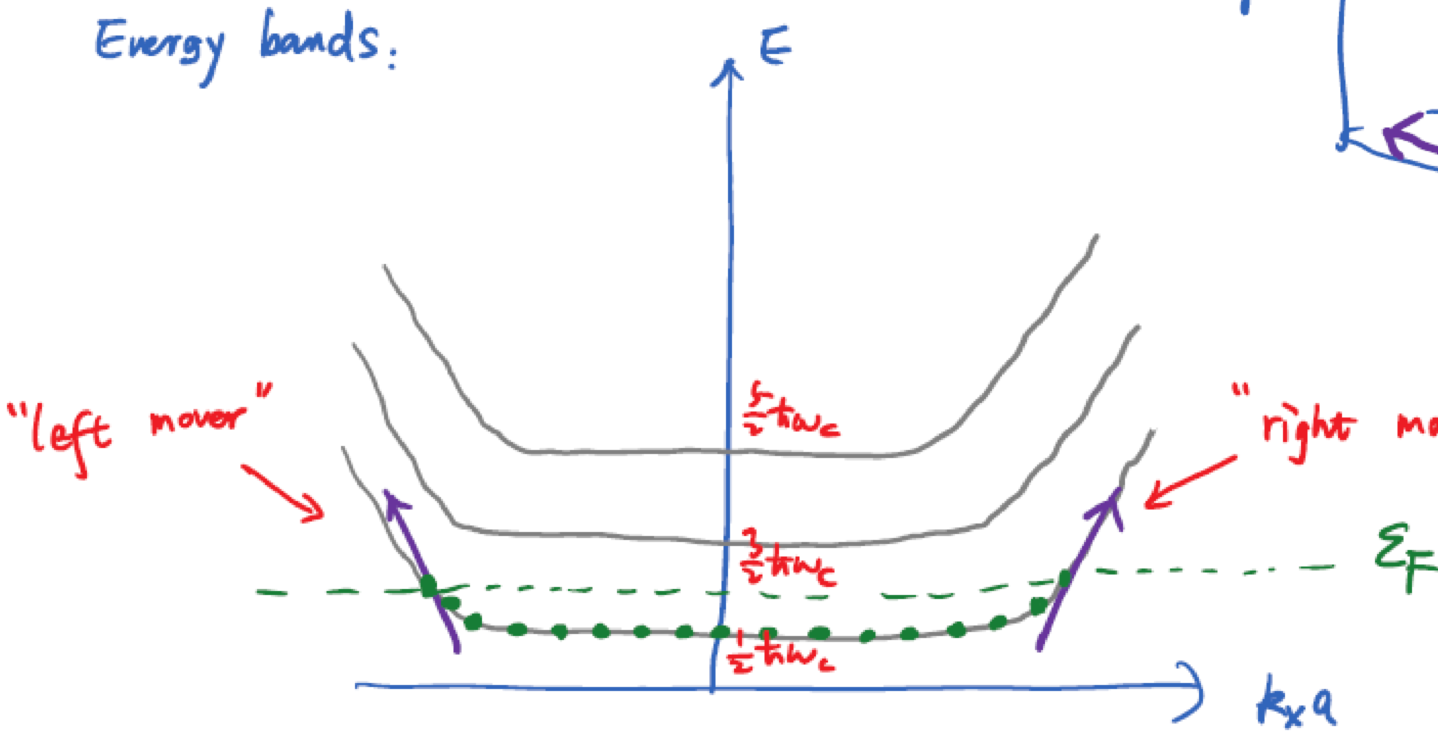
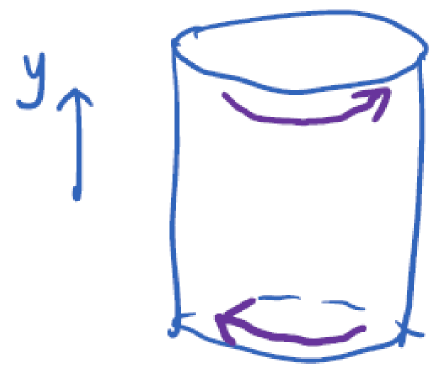
- Chiral edge states ("one-way movers"):





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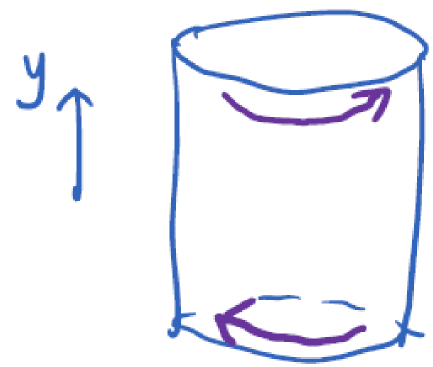

1) Landau levels

2) Chiral edge states at boundaries
   (crossing the Fermi energy)
   $\Downarrow$
   1D conducting channel in x-direction

$$H_R \simeq \sum_{k_x} \varepsilon_{k_x,R} \psi^\dagger_{k_x,R} \psi_{k_x,R} , \qquad \varepsilon_{k_x,R} \simeq v(k_x - k_F)$$

$$H_L \simeq \sum_{k_x} \varepsilon_{k_x,L} \psi^\dagger_{k_x,L} \psi_{k_x,L} , \qquad \varepsilon_{k_x,L} \simeq v(-k_x - k_F)$$

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- Further remarks:

1) chiral edge states exponentially localized at two boundaries of the cylinder $\Rightarrow$ This can be checked from the corresponding eigenvectors of $H(k_x)$.

2) Translation symmetry not important (disorder, impurities can be present) ← needs extra work to prove.

IQH: First example of "topological insulators"

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- Further remarks:

3) 1D chiral edge states cannot appear in pure 1D models with local terms on a lattice.

(Nielsen-Ninomiya theorem)

They can only appear as edge states of $D > 1$ topological insulators.

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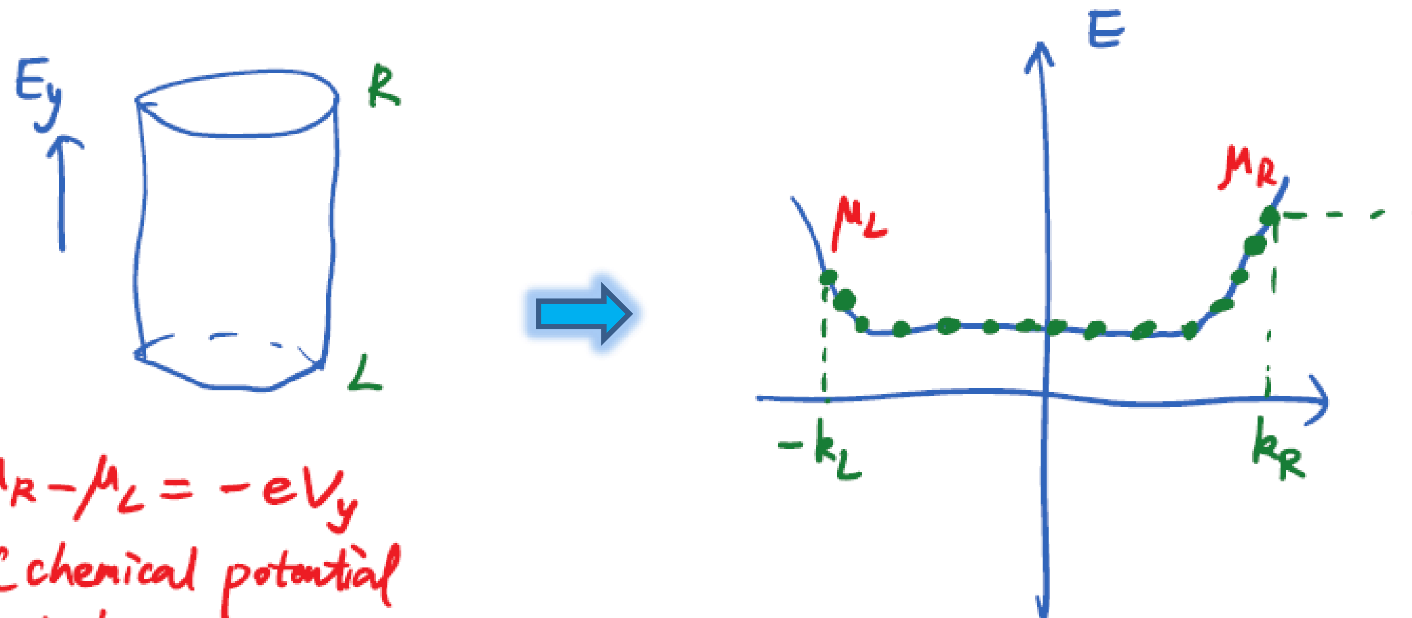
- Quantization of Hall conductance:



$\mu_R - \mu_L = -eV_y$

(chemical potential imbalanced due to external E_y)

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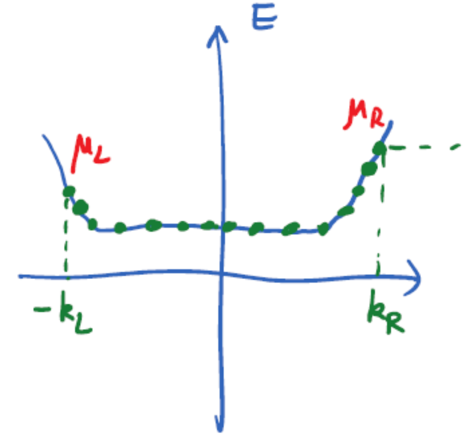
current: $I_x = (-e)\,\dfrac{1}{L_x} \sum\limits_{\text{occupied } k_x} v(k_x)$

$$= (-e) \int_{-k_L}^{k_R} \frac{dk_x}{2\pi}\, v(k_x)$$

$v(k_x) = \dfrac{1}{\hbar}\,\dfrac{dE(k_x)}{dk_x}$



$$= \frac{(-e)}{2\pi\hbar} \int_{\mu_L}^{\mu_R} dE$$

$2\pi\hbar = h$

$$= \frac{(-e)}{h}\,(\mu_R - \mu_L)$$

$(-e)V_y = \mu_R - \mu_L$

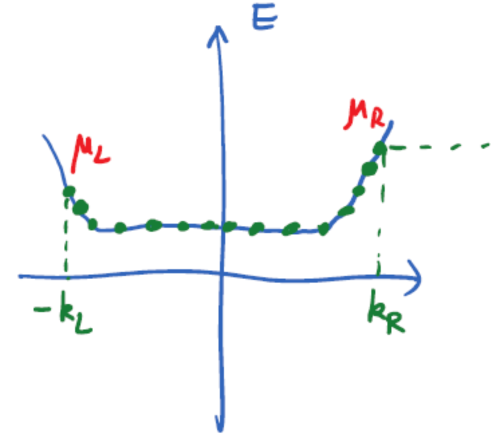
$$= \frac{e^2}{h}\, V_y$$

§ 5.2 Integer quantum Hall effect

current: $$I_x = (-e)\frac{1}{L_x}\sum_{\text{occupied } k_x} v(k_x)$$

$$= (-e)\int_{-k_L}^{k_R}\frac{dk_x}{2\pi}\, v(k_x) \qquad v(k_x) \rightarrow \frac{1}{\hbar}\frac{dE(k_x)}{dk_x}$$

$$= \frac{(-e)}{2\pi\hbar}\int_{\mu_L}^{\mu_R} dE \qquad 2\pi\hbar \rightarrow h$$



$$= \frac{(-e)}{h}(\mu_R - \mu_L) \qquad (\mu_R-\mu_L) \rightarrow (-e)V_y = \mu_R - \mu_L$$

$$= \frac{e^2}{h} V_y$$

$$\Rightarrow \boxed{\rho_{xy} = \frac{V_y}{I_x} = \frac{h}{e^2}}$$

(filling fraction $\nu = 1$)

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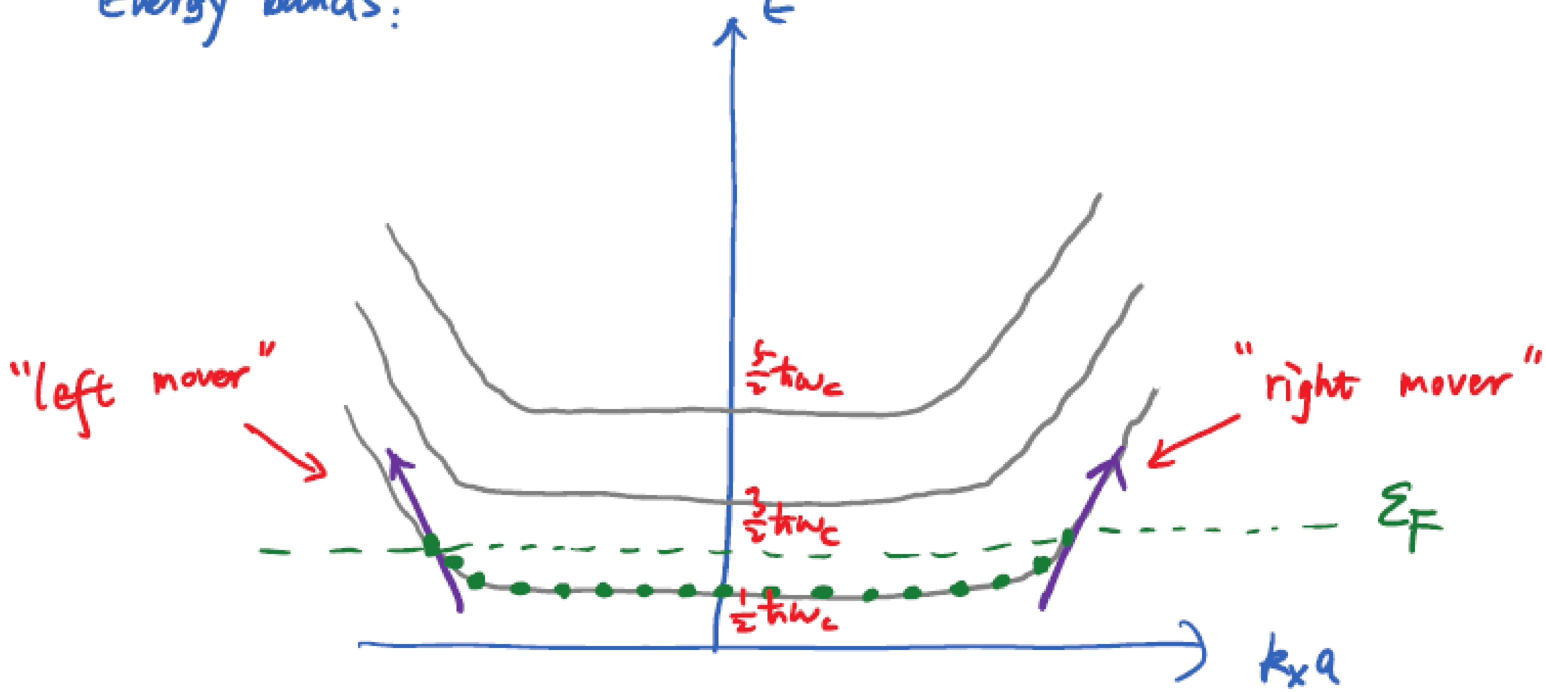
Energy bands:



n occupied Landau levels $\Rightarrow$ n pairs of chiral edge states

$$\Rightarrow \rho_{xy} = \frac{1}{n}\frac{h}{e^2}$$