SFB 1143

Solid State Theory (SS2020)

Lecture 19: Quantization of Hall conductance

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July 2nd, 2020

§ 5.3 Quantized Hall conductance

- Linear response theory & Kubo formula

$$H = H_0 + H'$$

electrons   weak external electric field

$$H' = e\vec{E}\cdot\vec{x} \qquad \vec{x} = \sum_{j=1}^{N}\vec{x}_j \quad (N: \text{\# of electrons})$$

Target: calculate $\sigma_{\mu\nu}$

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Ground state of H_0 : $|0\rangle$

$$H_0|0\rangle = \varepsilon_0|0\rangle$$

Assume GS unique !

Many-electron GS, NOT the usual "vacuum"!

Excited states of H_0 :

$$H_0|n\rangle = \varepsilon_n|n\rangle \quad , \quad n>0$$

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➢ Use non-degenerate perturbation theory to calculate perturbed ground state:

$$|\varphi\rangle = |0\rangle + \sum_{n>0} \frac{\langle n|H'|0\rangle}{\varepsilon_0 - \varepsilon_n} |n\rangle$$

Electric current:

$$\vec{J} = \frac{-e}{mS} \langle \varphi | \vec{\pi} | \varphi \rangle$$

area

mechanical momentum,

NOT canonical momentum!

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Electric current:

$$\vec{J} = \frac{-e}{mS} \langle \psi | \vec{\pi} | \psi \rangle$$

area

mechanical momentum,

NOT canonical momentum!

$$\left.\begin{aligned} \vec{\pi} &= m\dot{\vec{x}} \\ \underbrace{\dot{\vec{x}} = \frac{i}{\hbar}[H, \vec{x}]}_{\text{Heisenberg E.O.M.}} &= \frac{i}{\hbar}[H_0, \vec{x}] \end{aligned}\right\} \Rightarrow [\vec{x}, H_0] = \frac{i\hbar}{m}\vec{\pi}$$

$[H', \vec{x}] = 0$

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$$\Rightarrow \quad \vec{J} = -\frac{e}{mS}\left(\langle 0| + \sum_{n'>0} \frac{\langle 0|H'|n'\rangle}{\varepsilon_0 - \varepsilon_{n'}} \langle n'| \right) \vec{\pi}$$

$$\times \left(|0\rangle + \sum_{n>0} \frac{\langle n|H'|0\rangle}{\varepsilon_0 - \varepsilon_n} |n\rangle \right)$$

background current $\quad \vec{J}_0 = -\frac{e}{mS} \langle 0|\vec{\pi}|0\rangle = 0$

(no current without $\vec{E}$)

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$$\vec{J} = \frac{-e}{mS}\sum_{n>0}\frac{\langle 0|H'|n\rangle\langle n|\vec{\pi}|0\rangle + \langle n|H'|0\rangle\langle 0|\vec{\pi}|n\rangle}{\varepsilon_0-\varepsilon_n}$$

$$+ \underbrace{O(E^2)}$$

dropped as we consider linear response in $\vec{E}$!

$H' = e\vec{E}\cdot\vec{x}$

$$\simeq \frac{-e}{mS}\sum_{n>0}\frac{e\vec{E}\cdot\langle 0|\vec{x}|n\rangle\langle n|\vec{\pi}|0\rangle + e\vec{E}\cdot\langle n|\vec{x}|0\rangle\langle 0|\vec{\pi}|n\rangle}{\varepsilon_0-\varepsilon_n}$$

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$$\langle 0|\vec{x}|n\rangle = \frac{\langle 0|H_0\vec{x} - \vec{x}H_0|n\rangle}{\varepsilon_0 - \varepsilon_n}$$

$\langle 0|H_0 = \langle 0|\varepsilon_0$

$H_0|n\rangle = \varepsilon_n|n\rangle$

$$= \frac{\langle 0|[H_0, \vec{x}]|n\rangle}{\varepsilon_0 - \varepsilon_n}$$

$[H_0, \vec{x}] = -\frac{i\hbar}{m}\vec{\pi}$

$$= -\frac{i\hbar}{m}\frac{\langle 0|\vec{\pi}|n\rangle}{\varepsilon_0 - \varepsilon_n}$$

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$$\Rightarrow \vec{J} = \frac{ie^2\hbar}{m^2 S}\sum_{n>0} \frac{\vec{E}\cdot\langle 0|\vec{\pi}|n\rangle\langle n|\vec{\pi}|0\rangle - \vec{E}\cdot\langle n|\vec{\pi}|0\rangle\langle 0|\vec{\pi}|n\rangle}{(\varepsilon_0-\varepsilon_n)^2}$$

$$J_\mu = \frac{ie^2\hbar}{m^2 S}\sum_{n>0} \frac{\langle n|\pi^\mu|0\rangle\langle 0|\pi^\nu|n\rangle - \langle n|\pi^\nu|0\rangle\langle 0|\pi^\mu|n\rangle}{(\varepsilon_0-\varepsilon_n)^2} E_\nu$$

$$(\mu,\nu = x, y, z)$$

Compare with $J_\mu = \sum_\nu \sigma_{\mu\nu} E_\nu$!

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- Kubo formula for DC conductivity:

$$\sigma_{\mu\nu} = \frac{ie^2\hbar}{m^2 S} \sum_{n>0} \frac{\langle n|\pi_\mu|0\rangle\langle 0|\pi_\nu|n\rangle - \langle n|\pi_\nu|0\rangle\langle 0|\pi_\mu|n\rangle}{(\varepsilon_0 - \varepsilon_n)^2}$$

$$= \frac{e^2\hbar}{iS} \sum_{n>0} \frac{\langle 0|v_\mu|n\rangle\langle n|v_\nu|0\rangle - \langle 0|v_\nu|n\rangle\langle n|v_\mu|0\rangle}{(\varepsilon_0 - \varepsilon_n)^2}$$

velocity operator: $v_\mu = \dfrac{\pi_\mu}{m}$

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- Hall conductance as a topological invariant:

Landau gauge:
$\vec{A} = (0, Bx, 0)$

$$H = \sum_{j=1}^{N} \left[\frac{1}{2m_j} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} \right)^2 + \frac{1}{2m_j} \left(\frac{\hbar}{i} \frac{\partial}{\partial y_j} - eBx_j \right)^2 \right]$$

electrons may have different effective masses

$$+ \sum_{j=1}^{N} U(x_j, y_j) + \sum_{i<j} V(|\vec{r}_i - \vec{r}_j|)$$

impurity potential　　electron-electron interactions

D. J. Thouless, M. Kohmoto, M. P. Nightingale & M. den Nijs, PRL 49, 405 (1982);
Q. Niu, D. J. Thouless & Y. S. Wu, PRB 31, 3372 (1985).

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Velocity operators :

$$V_x = \sum_{j=1}^{N} \frac{1}{m_j} \cdot \underbrace{\frac{m_j}{i\hbar}[x_j, H_0]}_{\Pi_{j,x}} = \sum_{j=1}^{N} \frac{1}{m_j}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}\right)$$

$$V_y = \sum_{j=1}^{N} \frac{1}{m_j} \cdot \underbrace{\frac{m_j}{i\hbar}[y_j, H_0]}_{\Pi_{j,y}} = \sum_{j=1}^{N} \frac{1}{m_j}\left(\frac{\hbar}{i}\frac{\partial}{\partial y_j} - eBx_j\right)$$

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Hall conductance :

$$\sigma_{xy} = \frac{e^2\hbar}{iS} \sum_{n>0} \frac{\langle 0|v_x|n\rangle\langle n|v_y|0\rangle - \langle 0|v_y|n\rangle\langle n|v_x|0\rangle}{(\varepsilon_0 - \varepsilon_n)^2}$$

Not clear at this stage why σ_{xy} is quantized ...

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Define unitary transformation:

$$U = e^{-i\alpha \sum_{j=1}^{N} x_j} \, e^{-i\beta \sum_{j=1}^{N} y_j}$$

Note that $e^{-i\alpha x} \, p \, e^{i\alpha x} = p + \alpha\hbar$,

so U can shift momenta!

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Hamiltonian after unitary transformation:

$$\tilde{H}(\alpha,\beta) = U H U^{-1}$$

$$= \sum_{j=1}^{N}\left[\frac{1}{2m_j}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} + \alpha\hbar\right)^2 + \frac{1}{2m_j}\left(\frac{\hbar}{i}\frac{\partial}{\partial y_j} + \beta\hbar - eBx_j\right)^2\right]$$

$$+ \sum_{j=1}^{N} U(x_j, y_j) + \sum_{i<j} V(|\vec{r}_i - \vec{r}_j|)$$

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Velocity operators in the new basis:

$$\tilde{V}_x = U v_x U^{-1}$$

$$= \sum_{j=1}^{N} \frac{1}{m_j} \underbrace{U\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}\right)U^{-1}}$$

$$= \sum_{j=1}^{N} \frac{1}{m_j}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} + \alpha\hbar\right) \rightarrow \frac{\partial \tilde{H}(\alpha,\beta)}{\partial\alpha} = \hbar\tilde{V}_x$$

$$\tilde{V}_y = U v_y U^{-1}$$

$$= \sum_{j=1}^{N} \frac{1}{m_j}\left(\frac{\hbar}{i}\frac{\partial}{\partial y_j} + \beta\hbar - eBx_j\right) \rightarrow \frac{\partial \tilde{H}(\alpha,\beta)}{\partial\beta} = \hbar\tilde{V}_y$$

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σ_{xy} in the new basis :

$$\sigma_{xy} = \frac{e^2\hbar}{iS} \sum_{n>0} \frac{\langle 0|\tilde{V}_x|n\rangle\langle n|\tilde{V}_y|0\rangle - \langle 0|\tilde{V}_y|n\rangle\langle n|\tilde{V}_x|0\rangle}{(\varepsilon_0 - \varepsilon_n)^2}$$

$$= \frac{e^2}{iS\hbar} \sum_{n>0} \frac{\langle 0|\frac{\partial \tilde{H}}{\partial \alpha}|n\rangle\langle n|\frac{\partial \tilde{H}}{\partial \beta}|0\rangle - \langle 0|\frac{\partial \tilde{H}}{\partial \beta}|n\rangle\langle n|\frac{\partial \tilde{H}}{\partial \alpha}|0\rangle}{(\varepsilon_0 - \varepsilon_n)^2}$$

Remark : $|0\rangle$ and $|n\rangle$ now depend on α and β !

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$$\langle 0|\frac{\partial \tilde{H}}{\partial \alpha}|n\rangle\langle n|\frac{\partial \tilde{H}}{\partial \beta}|0\rangle$$

$$= \left(\frac{\partial}{\partial \alpha} \underbrace{\langle 0|\tilde{H}|n\rangle}_{\varepsilon_0\langle 0|n\rangle = 0} - \underbrace{\langle \tfrac{\partial}{\partial \alpha} 0|\tilde{H}|n\rangle}_{\varepsilon_n \langle \frac{\partial}{\partial \alpha} 0|n\rangle} - \underbrace{\langle 0|\tilde{H}|\tfrac{\partial}{\partial \alpha} n\rangle}_{\varepsilon_0\langle 0|\frac{\partial}{\partial \alpha} n\rangle = -\varepsilon_0 \langle \frac{\partial}{\partial \alpha} 0|n\rangle} \right)$$

$$\times \left(\frac{\partial}{\partial \beta} \underbrace{\langle n|\tilde{H}|0\rangle}_{0} - \underbrace{\langle \tfrac{\partial}{\partial \beta} n|\tilde{H}|0\rangle}_{\varepsilon_0 \langle \frac{\partial}{\partial \beta} n|0\rangle = -\varepsilon_0 \langle n|\frac{\partial}{\partial \beta} 0\rangle} - \underbrace{\langle n|\tilde{H}|\tfrac{\partial}{\partial \beta} 0\rangle}_{\varepsilon_n \langle n|\frac{\partial}{\partial \beta} 0\rangle} \right)$$

$$= (\varepsilon_0 - \varepsilon_n)^2 \langle \frac{\partial}{\partial \alpha} 0|n\rangle\langle n|\frac{\partial}{\partial \beta} 0\rangle$$

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$$\sigma_{xy} = \frac{e^2}{iS\hbar} \sum_{n>0} \frac{\langle 0|\frac{\partial \tilde{H}}{\partial \alpha}|n\rangle\langle n|\frac{\partial \tilde{H}}{\partial \beta}|0\rangle - \langle 0|\frac{\partial \tilde{H}}{\partial \beta}|n\rangle\langle n|\frac{\partial \tilde{H}}{\partial \alpha}|0\rangle}{(\varepsilon_0 - \varepsilon_n)^2}$$

$$= \frac{e^2}{iS\hbar} \sum_{n>0} \left[\langle \tfrac{\partial}{\partial \alpha} 0 | n\rangle\langle n | \tfrac{\partial}{\partial \beta} 0\rangle - \langle \tfrac{\partial}{\partial \beta} 0 | n\rangle\langle n | \tfrac{\partial}{\partial \alpha} 0\rangle \right]$$

$$\sum_{n>0} |n\rangle\langle n| = 1 - |0\rangle\langle 0|$$

$$= \frac{e^2}{iS\hbar} \left(\langle \tfrac{\partial}{\partial \alpha} 0 | \tfrac{\partial}{\partial \beta} 0\rangle - \langle \tfrac{\partial}{\partial \beta} 0 | \tfrac{\partial}{\partial \alpha} 0\rangle \right)$$

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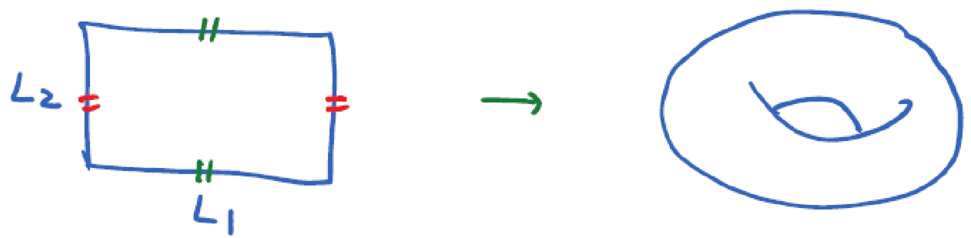
Define $\theta = \alpha L_1$, $\varphi = \beta L_2$



"twist boundary condition",

parametrized by two angles $\theta \in [0, 2\pi)$,
$\varphi \in [0, 2\pi)$.

$$\sigma_{xy} = \frac{e^2}{i\hbar}\left(\langle \frac{\partial}{\partial\theta} 0 | \frac{\partial}{\partial\varphi} 0 \rangle - \langle \frac{\partial}{\partial\varphi} 0 | \frac{\partial}{\partial\theta} 0 \rangle \right)$$

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We expect that σ_{xy} should be insensitive to boundary conditions for L_1, L_2 large and $N \to \infty$.

Thus, we would average σ_{xy} over all possible twisted boundary conditions $\theta \in [0, 2\pi)$, $\varphi \in [0, 2\pi)$:

$$\sigma_{xy} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\theta \, d\varphi \, \frac{e^2}{i\hbar} \left(\langle \frac{\partial}{\partial \theta} 0 | \frac{\partial}{\partial \varphi} 0 \rangle - \langle \frac{\partial}{\partial \varphi} 0 | \frac{\partial}{\partial \theta} 0 \rangle \right)$$

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$$\sigma_{xy} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} d\theta\, d\varphi\, \frac{e^2}{i\hbar}\left(\langle \frac{\partial}{\partial\theta} 0 | \frac{\partial}{\partial\varphi} 0 \rangle - \langle \frac{\partial}{\partial\varphi} 0 | \frac{\partial}{\partial\theta} 0 \rangle \right)$$

$$= \frac{e^2}{h}\int_0^{2\pi}\int_0^{2\pi} d\theta\, d\varphi\, \frac{1}{2\pi i}\left(\langle \frac{\partial}{\partial\theta} 0 | \frac{\partial}{\partial\varphi} 0 \rangle - \langle \frac{\partial}{\partial\varphi} 0 | \frac{\partial}{\partial\theta} 0 \rangle \right)$$

topological invariant !

(The first Chern number $\Rightarrow$ integers)

Non-perturbative proof for the quantization of σ_{xy}