TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

SFB 1143

Tensor Networks (SS2021)

Lecture 14: Symmetry in MPS

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June 21st, 2021

§ 1.8 Symmetric MPS

Q: How can we parametrize an MPS such that it is invariant (or transforms covariantly) under a symmetry?

$$U_g|\psi\rangle = e^{i\theta_g}|\psi\rangle$$

Motivation:

- Numerical purpose (accelerate calculations, achieve larger bond dimensions…)
- Classification of phases under symmetries

§ 1.8 Symmetric MPS

Example: spin-1/2 Heisenberg chain

$$H = \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1}$$

$$\left[H, \sum_{i=1}^{N} S_i^a\right] = 0 \qquad (a = x, y, z)$$

⇨ SU(2) symmetry

$$U_g H U_g^\dagger = H \qquad g \in \mathrm{SU}(2)$$

Parametrization with Euler angles:

$$U_{(\alpha,\beta,\gamma)} = e^{-i\alpha \sum_{i=1}^{N} S_i^z} e^{-i\beta \sum_{i=1}^{N} S_i^y} e^{-i\gamma \sum_{i=1}^{N} S_i^z} = u_{(\alpha,\beta,\gamma)}^{\otimes N}$$

§ 1.8 Symmetric MPS

Example: spin-1/2 Heisenberg chain

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$$\left[H, \sum_{i=1}^{N} S_i^a\right] = 0 \qquad (a = x, y, z)$$ ⇨ SU(2) symmetry

$$U_g H U_g^\dagger = H \qquad g \in \mathrm{SU}(2)$$

Ground state (even N): SU(2) singlet

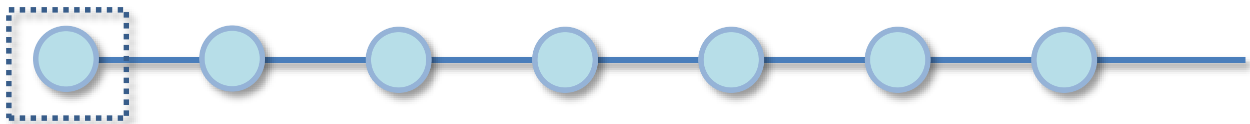
$$U_g|\psi\rangle = |\psi\rangle$$

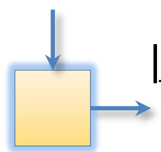
§ 1.8 Symmetric MPS

- Symmetry imposes constraints on MPS matrices, which can be illustrated with a real-space RG…

§ 1.8 Symmetric MPS

$S = 1/2$





$$|J_1, \alpha_1\rangle = \sum_{m_1=-S}^{S} A_{(J_1,\alpha_1)}^{[S,m_1]} |S, m_1\rangle$$

$J_1 = 1/2$

$$\delta_{m_1,\alpha_1} = \langle S, m_1 | J_1, \alpha_1 \rangle$$

§ 1.8 Symmetric MPS

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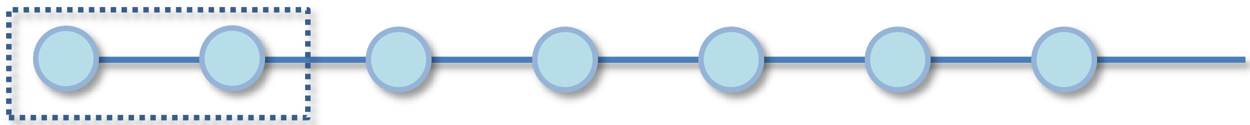
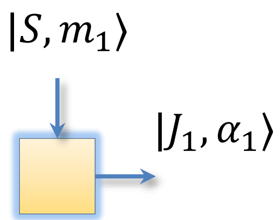
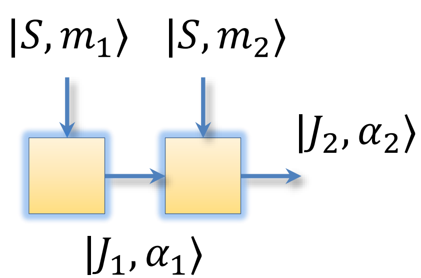




$$|J_1, \alpha_1\rangle = \sum_{m_1=-S}^{S} A^{[S,m_1]}_{(J_1,\alpha_1)} |S, m_1\rangle$$

$J_1 = 1/2$



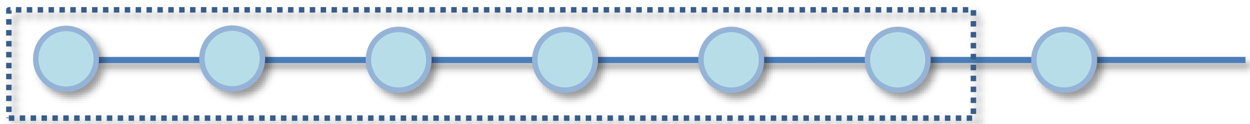
$J_2 = 0, 1$

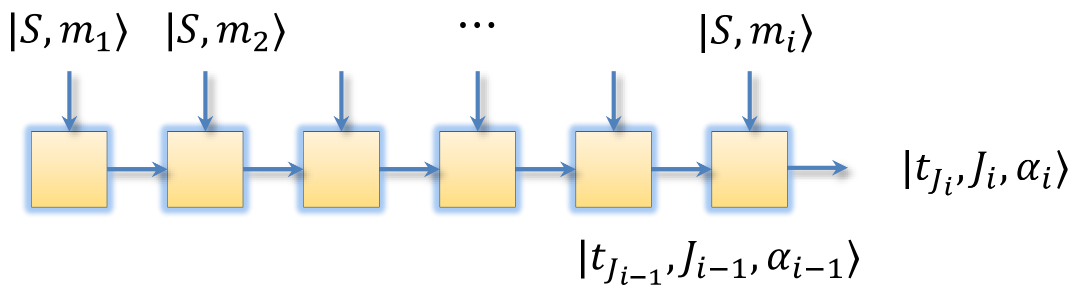
$$|J_2, \alpha_2\rangle = \sum_{(J_1,\alpha_1),(S,m_2)} A^{[S,m_2]}_{(J_1,\alpha_1);(J_2,\alpha_2)} |J_1, \alpha_1\rangle \otimes |S, m_2\rangle$$

Clebsch-Gordon coefficient: $\langle J_1, \alpha_1; S, m_2 | J_2, \alpha_2\rangle$

§ 1.8 Symmetric MPS

$S = 1/2$





$$|t_{J_i}, J_i, \alpha_i\rangle = \sum_{(t_{J_{i-1}},J_{i-1},\alpha_{i-1}),(S,m_i)} A^{[S,m_i]}_{(t_{J_{i-1}},J_{i-1},\alpha_{i-1});(t_{J_i},J_i,\alpha_i)} \, |t_{J_{i-1}}, J_{i-1}, \alpha_{i-1}\rangle \otimes |S, m_i\rangle$$

t_{J_i} distinguish different states with the same J_i .

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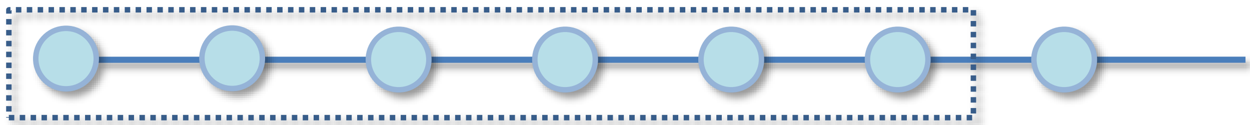
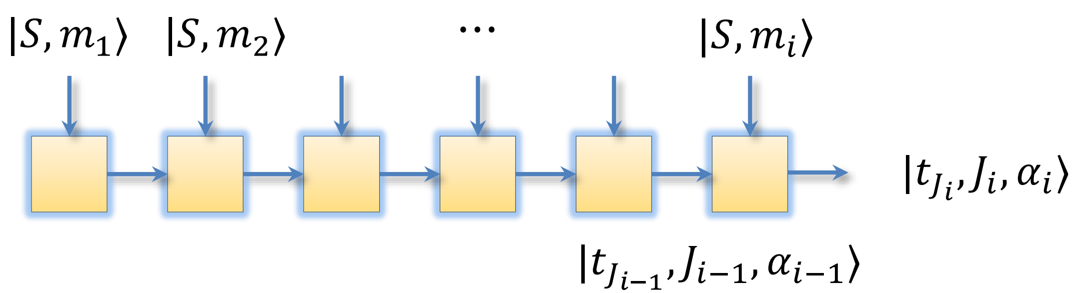
$S = 1/2$



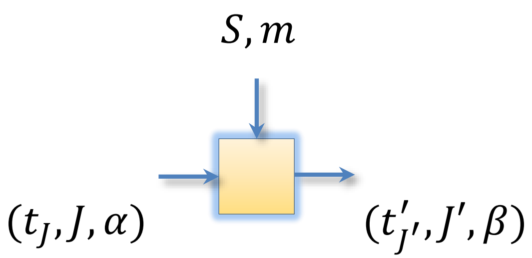


$$|t_{J_i}, J_i, \alpha_i\rangle = \sum_{(t_{J_{i-1}}, J_{i-1}, \alpha_{i-1}),(S, m_i)} \underline{A^{[S,m_i]}_{(t_{J_{i-1}}, J_{i-1}, \alpha_{i-1});(t_{J_i}, J_i, \alpha_i)}} \, |t_{J_{i-1}}, J_{i-1}, \alpha_{i-1}\rangle \otimes |S, m_i\rangle$$

$$\downarrow$$

$$T_{(t_{J_{i-1}}, J_{i-1});(t_{J_i}, J_i)} \langle J_{i-1}, \alpha_{i-1}; S, m_i | J_i, \alpha_i\rangle$$

§ 1.8 Symmetric MPS



$$A^{(S,m)}_{(t_J,J,\alpha);(t'_{J'},J',\beta)} = T_{(t_J,J);(t'_{J'},J')} \langle J,\alpha;S,m|J',\beta\rangle$$

"structure" tensor
(not fixed by symmetry), encoding variational parameters

Clebsch-Gordon coefficient
(fixed by symmetry)

Example: spin-1 AKLT state ($S = 1, J = J' = 1/2$, structure tensor not needed)

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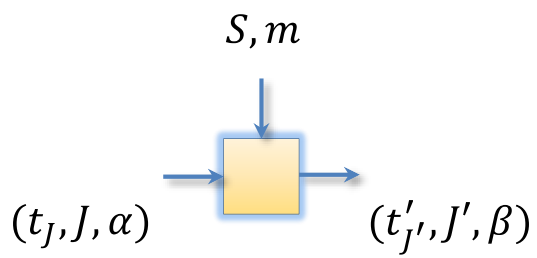


$$A^{(S,m)}_{(t_J,J,\alpha);(t'_{J'},J',\beta)} = \underbrace{T_{(t_J,J);(t'_{J'},J')}}\,\underbrace{\langle J,\alpha;S,m|J',\beta\rangle}$$

"structure" tensor
(not fixed by symmetry), encoding variational parameters

Clebsch-Gordon coefficient
(fixed by symmetry)

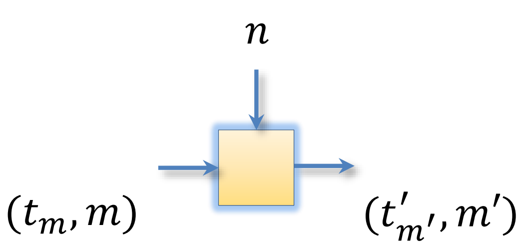
See, e.g., P. Schmoll, S. Singh, M. Rizzi & R. Orús, Ann. Phys. 419, 168232(2020).

§ 1.8 Symmetric MPS

- What we worked out is the parametrization of MPS matrices (three-leg tensors) with the non-Abelian SU(2) symmetry.

- In fact, Abelian symmetries, such as U(1) and $\mathbb{Z}_2$, are simpler and more commonly used in numerics.

Example: Particle-number conservation [U(1) symmetry]



$$A^n_{(t_m,m);(t'_{m'},m')} = T_{t_m,t'_{m'}} \delta_{m+n,m'}$$

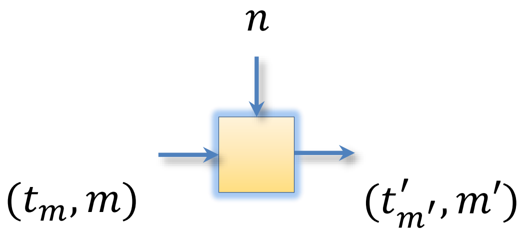
$$n, m, m' = \begin{cases} 0,1 & \text{for fermion} \\ 0,1,2,\ldots & \text{for boson} \end{cases}$$

§ 1.8 Symmetric MPS

- What we worked out is the parametrization of MPS matrices (three-leg tensors) with the non-Abelian SU(2) symmetry.

- In fact, Abelian symmetries, such as U(1) and $\mathbb{Z}_2$, are simpler and more commonly used in numerics.

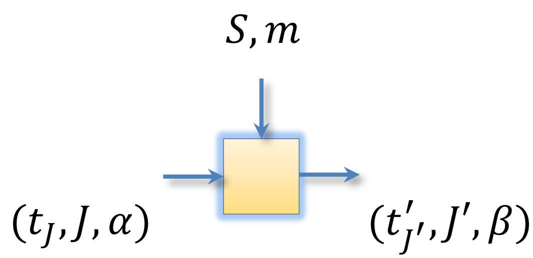
Example: Fermion parity conservation ($\mathbb{Z}_2$ symmetry)



$$A^n_{(t_m,m);(t'_{m'},m')} = T_{t_m,t'_{m'}} \delta_{m+n,m' \pmod 2}$$

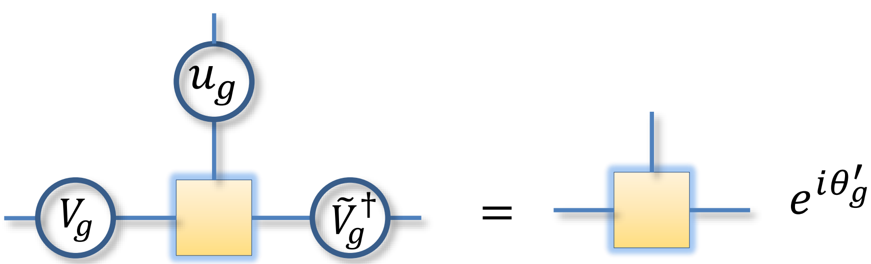
$n, m, m' = 0,1$
($\mathbb{Z}_2$ charges)

§ 1.8 Symmetric MPS



$$A^{(S,m)}_{(t_J,J,\alpha);(t'_{J'},J',\beta)} = T_{(t_J,J);(t'_{J'},J')} \langle J, \alpha; S, m | J', \beta \rangle$$

More abstract way to think about this: Schur's lemma



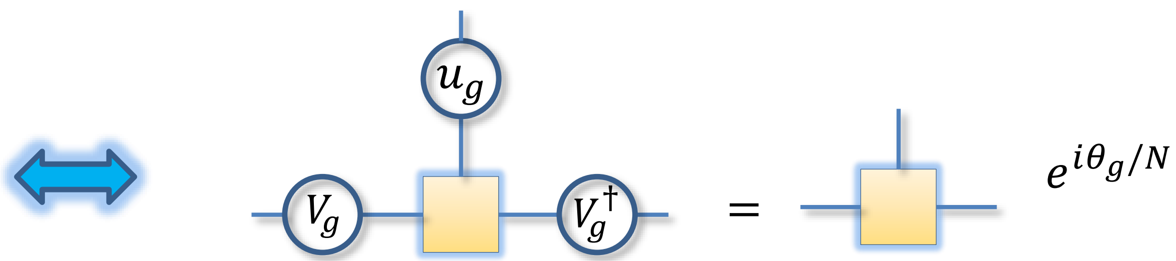
$$V_g = \begin{pmatrix} v_g & & \\ & v'_g & \\ & & \ddots \end{pmatrix}$$

§ 1.8 Symmetric MPS

- Consider an on-site unitary symmetry group G and a translationally invariant MPS.

- Sufficient and necessary condition for an MPS to be invariant under the symmetry action of G:

$$U_g|\psi\rangle = e^{i\theta_g}|\psi\rangle$$

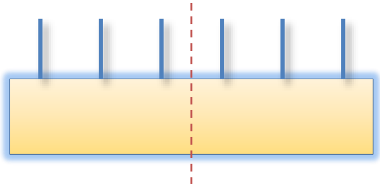


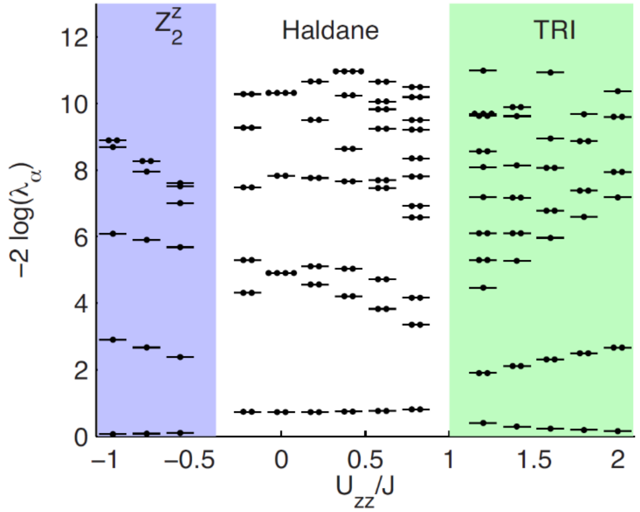
D. Perez-Garcia, M. Wolf, M. Sanz, F. Verstraete & J.I. Cirac, Phys. Rev. Lett. 100, 167202 (2008).

§ 1.9 Symmetry-protected topological phase

- Let us use this symmetry condition to understand the protected degeneracy in the entanglement spectrum of spin-1 Haldane chains (this uncovers the symmetry-protected topological phases).

$$H_0 = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + U_{zz} \sum_j (S_j^z)^2$$





F. Pollmann, A. M. Turner, E. Berg & M. Oshikawa, Phys. Rev. B 81, 064439 (2010).

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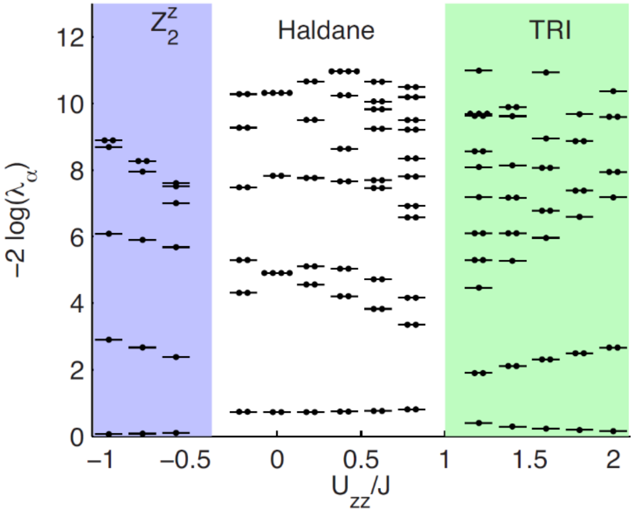

Onsite unitary symmetry: $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$u_g = \{I, e^{i\pi S^z}\} \times \{I, e^{i\pi S^x}\}$$

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