TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

SFB 1143

Tensor Networks (SS2021)

Lecture 15: Symmetry-protected topological phase

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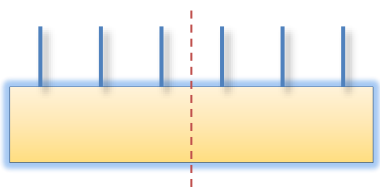
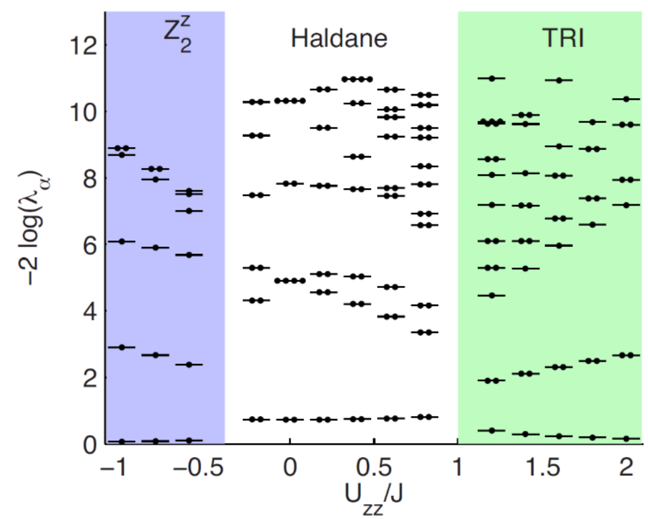
§ 1.9 Symmetry-protected topological phase

- Let us use the symmetry condition to understand the protected degeneracy in the entanglement spectrum of spin-1 Haldane chains (this uncovers the symmetry-protected topological phases).

$$H_0 = J\sum_j \vec{S}_j \cdot \vec{S}_{j+1} + U_{zz}\sum_j (S_j^z)^2$$





F. Pollmann, A. M. Turner, E. Berg & M. Oshikawa, Phys. Rev. B 81, 064439 (2010).

§ 1.9 Symmetry-protected topological phase

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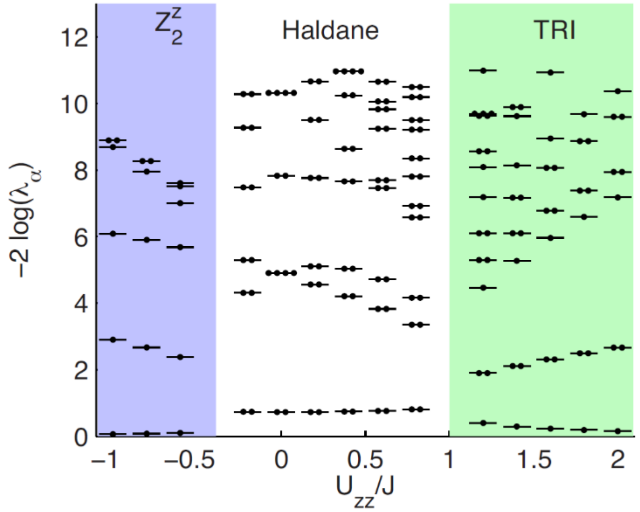
Onsite unitary symmetry: $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$u_g = \{I, e^{i\pi S^z}\} \times \{I, e^{i\pi S^x}\}$$



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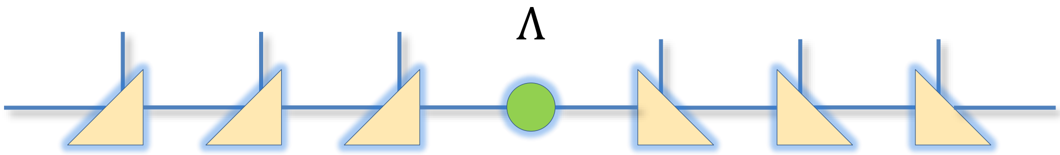
§ 1.9 Symmetry-protected topological phase

- The spin-1 Haldane phase is well described by MPS. Think of the Schmidt decomposition of the MPS:



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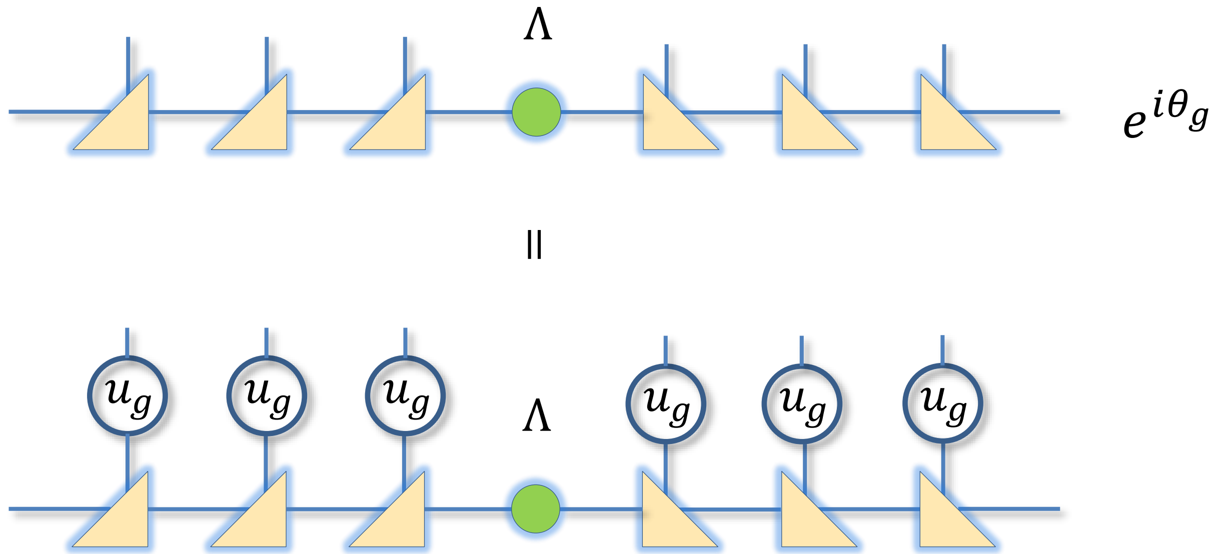

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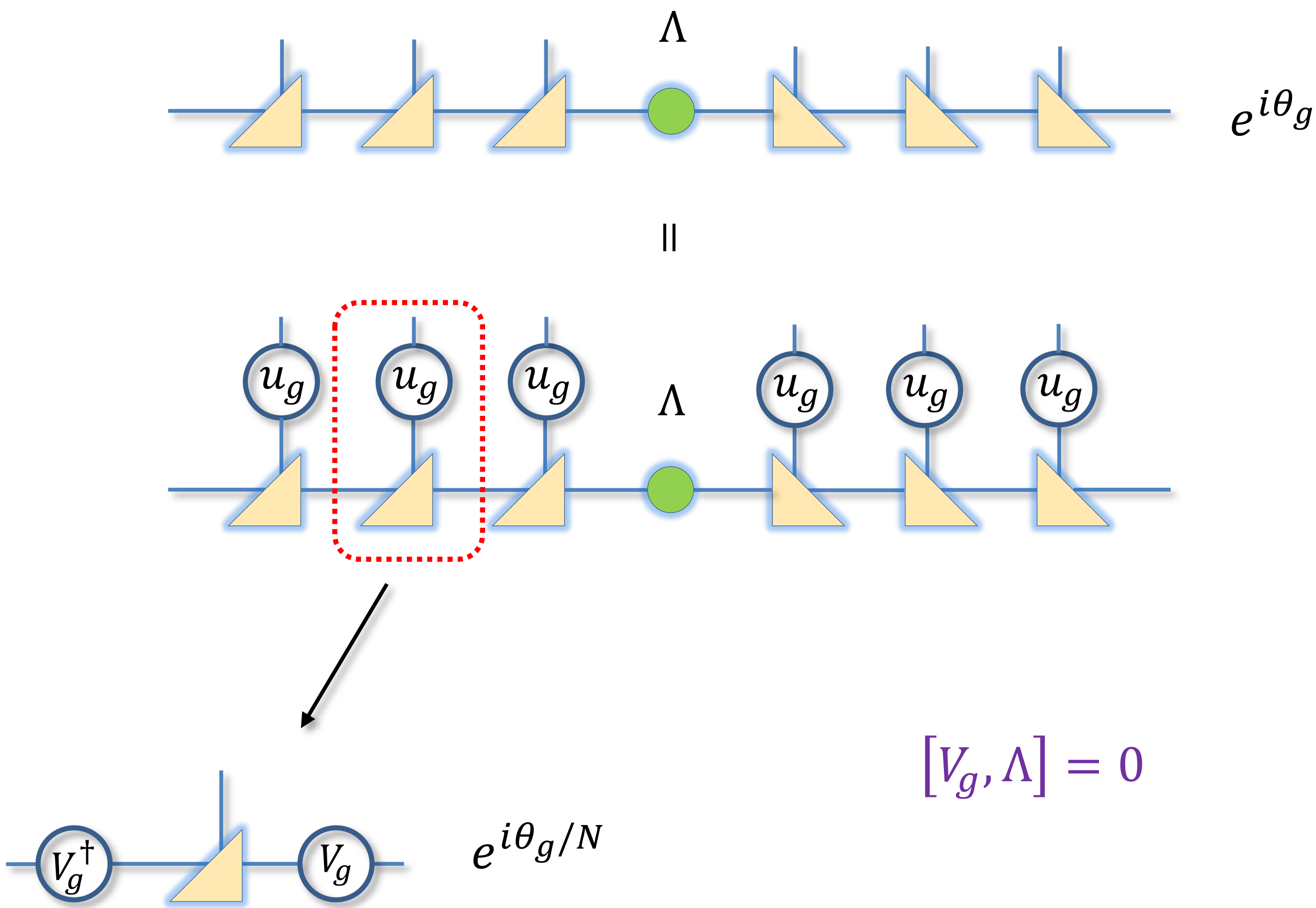
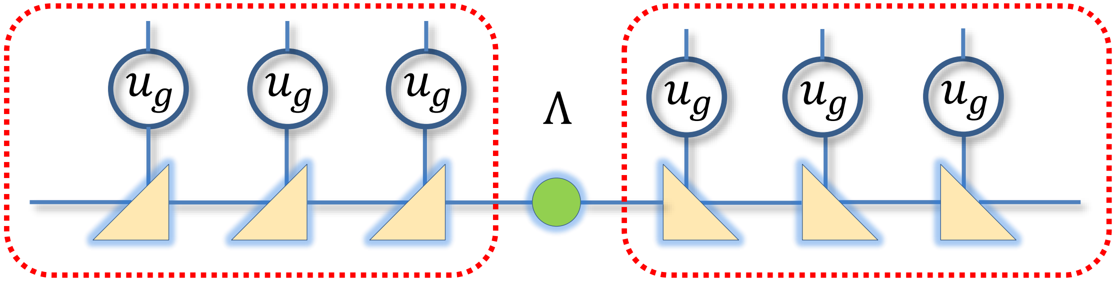


$$[V_g, \Lambda] = 0$$

§ 1.9 Symmetry-protected topological phase



Symmetry transformation on Schmidt vectors:

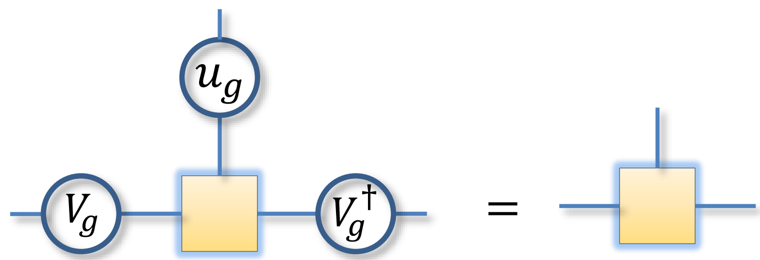
$$\otimes_{i\in L}\, u_g^{(i)}|\phi_\alpha^L\rangle = \sum_\beta \left(V_g^\dagger\right)_{\beta\alpha}|\phi_\beta^L\rangle$$

(phase ignored)

$$\otimes_{i\in R}\, u_g^{(i)}|\phi_\alpha^R\rangle = \sum_\gamma \left(V_g\right)_{\alpha\gamma}|\phi_\gamma^R\rangle$$

§ 1.9 Symmetry-protected topological phase

- What is V_g?



Remark: V_g has a phase ambiguity ($V_g \rightarrow e^{i\phi_g} V_g$)!

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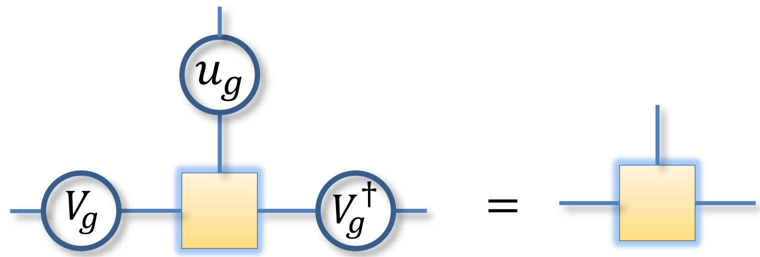

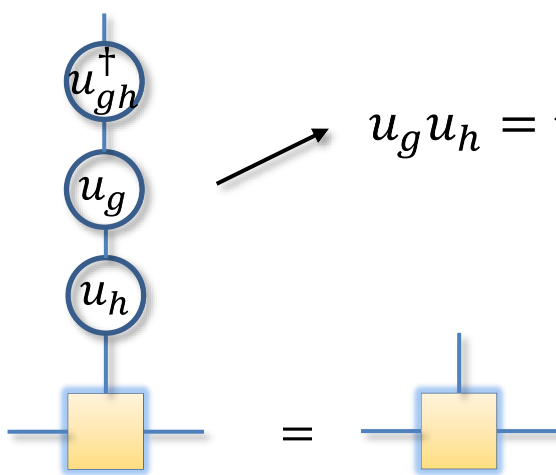

$u_g u_h = u_{gh}$ linear representation of G

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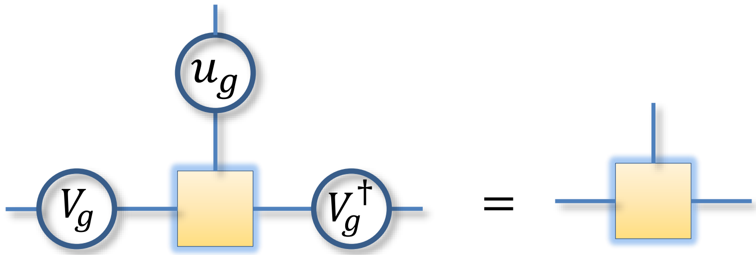

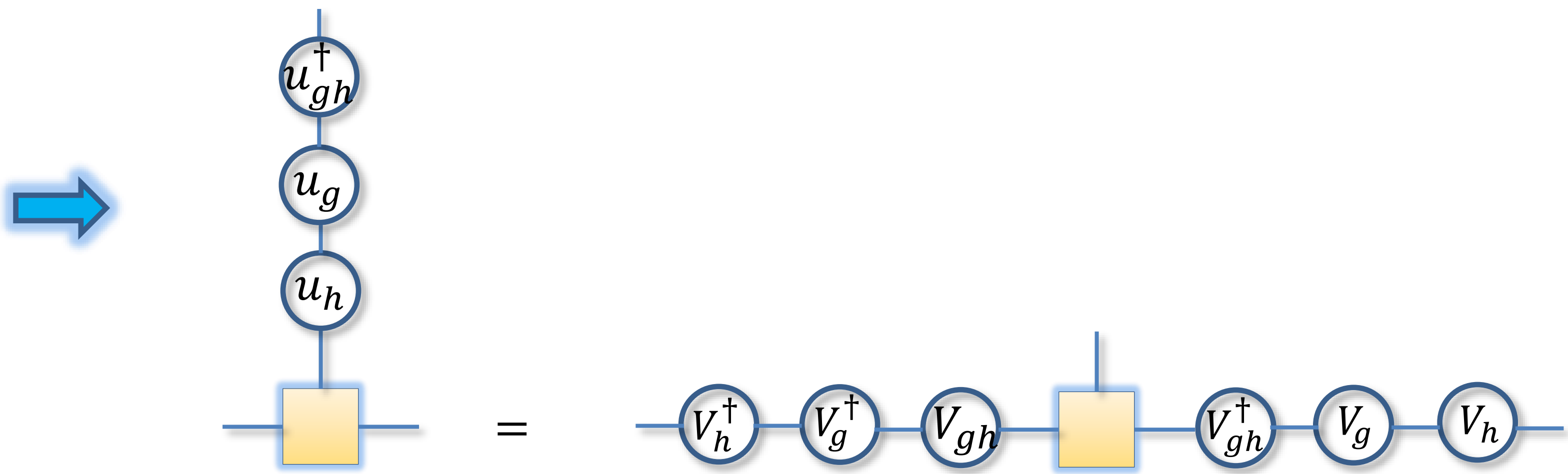

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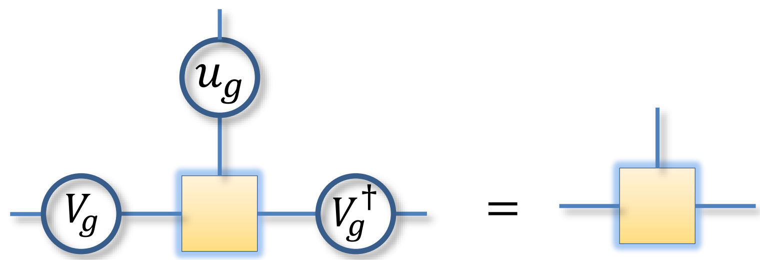

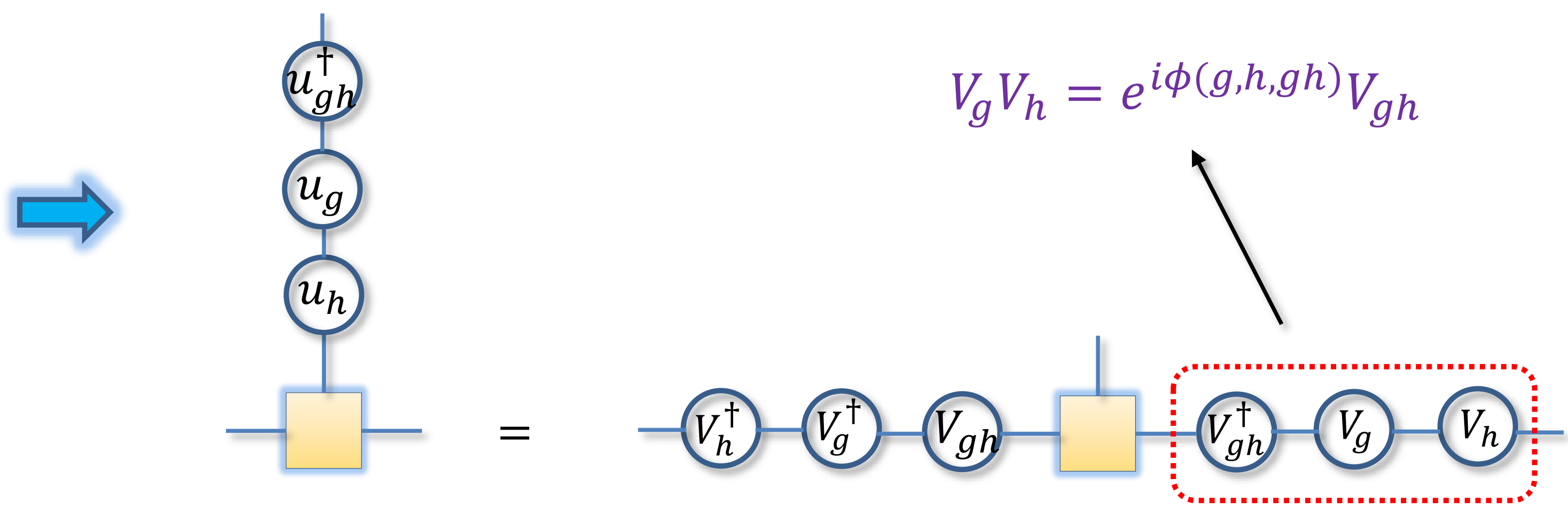

§ 1.9 Symmetry-protected topological phase

- Linear representation vs. projective representation:

Group G: $gh = f$ → $u_g u_h = u_f$

Group G: $gh = f$ → $V_g V_h = e^{i\phi(g,h,f)} V_f$

Remark: Equivalence class of $e^{i\phi(g,h,gh)}$ defines projective representations. (Note that $e^{i\phi(g,h,f)}$ itself is not invariant under $V_g \to e^{i\theta_g} V_g$!)

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Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with $g \in \{I, X, Y, Z\}$

Multiplication table:

$$X^2 = Y^2 = Z^2 = I \qquad XY = YX = Z$$

$$YZ = ZY = X \qquad ZX = XZ = Y$$

Nontrivial projective representation:

$$V_X^2 = V_Y^2 = V_Z^2 = I \qquad V_X V_Y = -V_Y V_Z = iV_Z$$

$$V_Y V_Z = -V_Z V_Y = iV_X \qquad V_Z V_X = -V_X V_Z = iV_Y$$

Example: Pauli matrices!

§ 1.9 Symmetry-protected topological phase

- Non-trivial projective representations protect the degeneracy in the entanglement spectrum.

$$[V_g, \Lambda] = 0 \qquad V_g V_h = e^{i\phi(g,h,gh)} V_{gh}$$

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- Non-trivial projective representations protect the degeneracy in the entanglement spectrum.

$$[V_g, \Lambda] = 0 \qquad V_g V_h = e^{i\phi(g,h,gh)} V_{gh}$$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry in spin-1 chains

$$[V_X, \Lambda] = [V_Z, \Lambda] = 0$$

$$V_X V_Z = \pm V_Z V_X$$

The sign plays the role of an order parameter!

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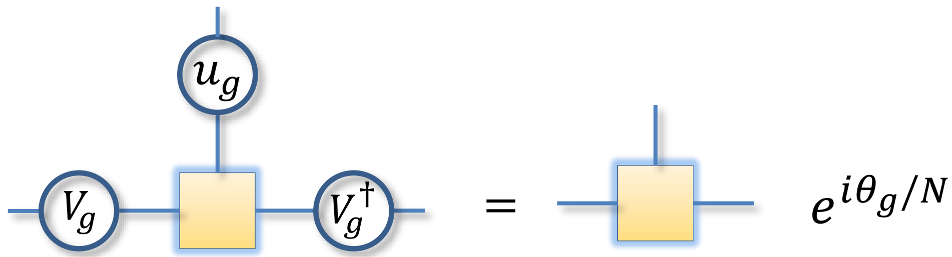
How can we check/distinguish projective representations from MPS?



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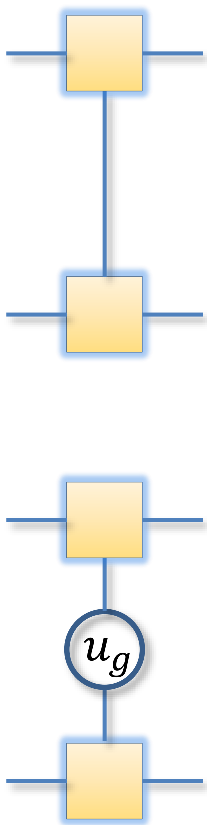

left/right canonical form

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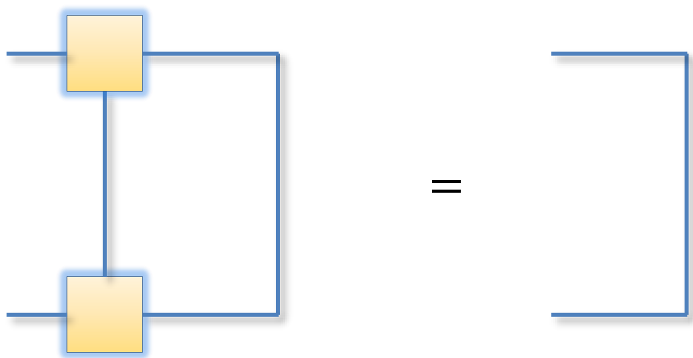
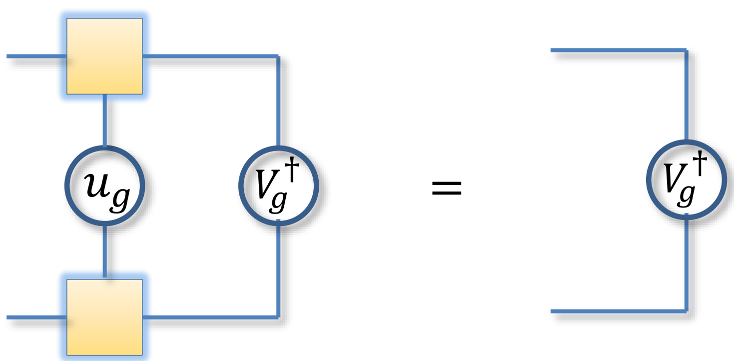


Leading eigenvector in the right canonical form



$e^{i\theta_g/N}$

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How can we check/distinguish projective representations from MPS?

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry in spin-1 chains

$$V_X V_Z = \pm V_Z V_X$$

$$\Rightarrow \quad \frac{1}{D} \mathrm{tr}(V_Z^\dagger V_X V_Z V_X^\dagger) = \pm 1$$

F. Pollmann & A. M. Turner, Phys. Rev. B 86, 125441 (2012);

J. Haegeman, D. Pérez-García, I. Cirac & N. Schuch, Phys. Rev. Lett. 109, 050402 (2012).

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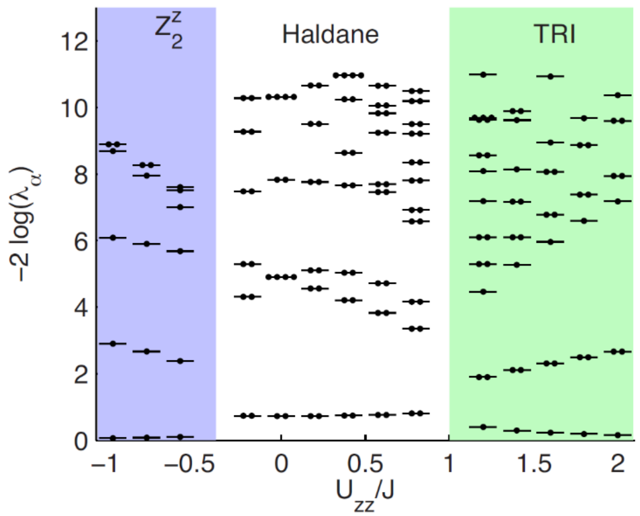
- For the spin-1 Haldane chains, the two-fold degeneracy in the entanglement spectrum is protected by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, bond-centered inversion symmetry, and time-reversal symmetry.

$$H_0 = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + U_{zz} \sum_j (S_j^z)^2$$



- ➢ Inversion and time-reversal symmetries will be discussed in Tutorial 6.

F. Pollmann, A. M. Turner, E. Berg & M. Oshikawa, Phys. Rev. B 81, 064439 (2010).