TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

SFB 1143

Tensor Networks (SS2021)

Lecture 2: Matrix Product State from real-space renormalization

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April 15th, 2021

§ 0. Tensor Network Basics

Idea of tensor networks: factorizing large tensors into smaller pieces

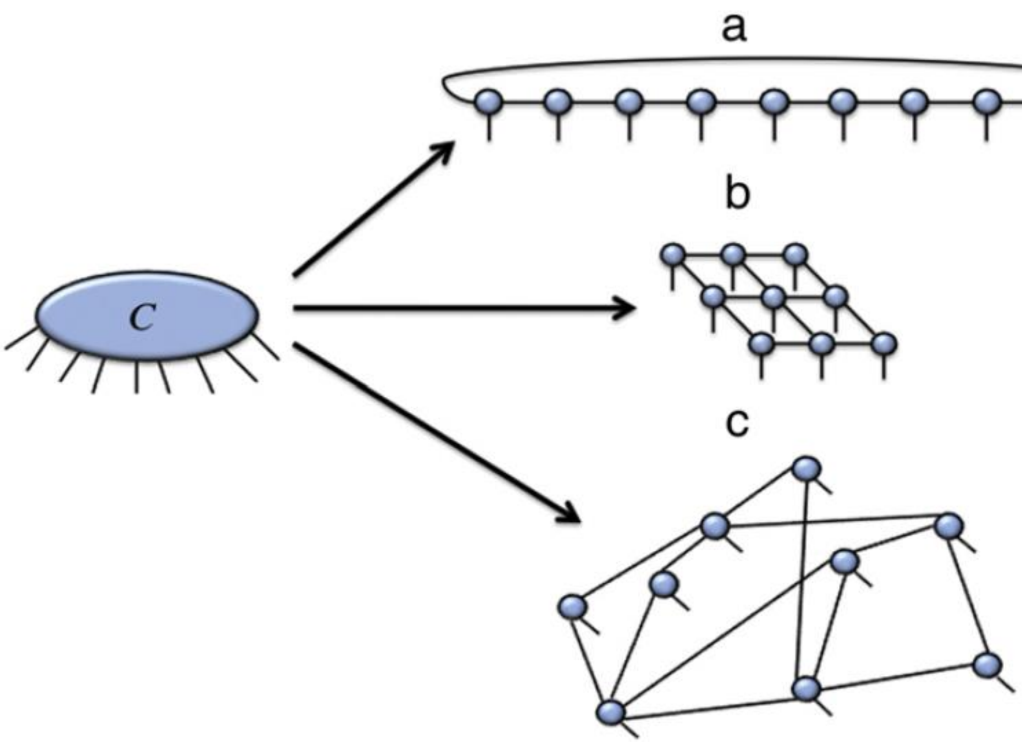


Figure from R. Orús, Ann. Phys. 349, 117 (2014)

§ 0. Tensor Network Basics

Tensor networks have applications in many fields:

- Condensed matter physics
- Quantum chemistry
- Quantum information and computation
- High energy physics
- Applied mathematics
- Machine learning
- …

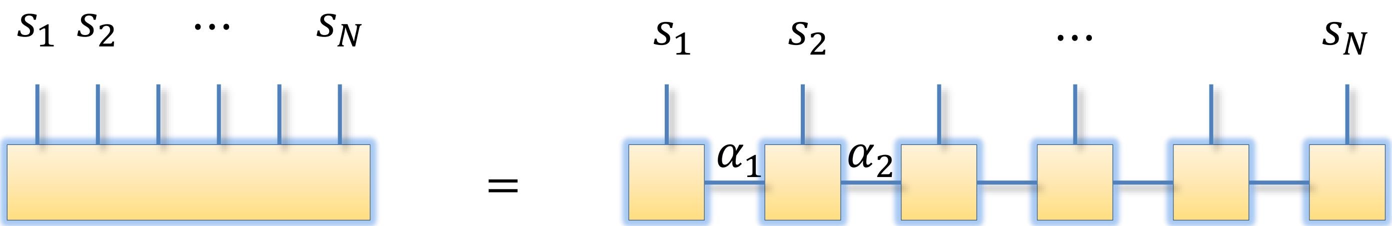
§ 0. Tensor Network Basics

Many natural and interesting questions:

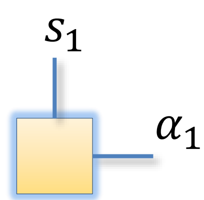
- How to justify whether tensor networks are good approximations?
- Which tensor network works better for what systems?
- How to optimize the local tensors for a given Hamiltonian?
- Classification of quantum phases of matter?
- Relation between quantum entanglement and emergent space-time?
- …

§ 1. Matrix Product State

Matrix Product State:



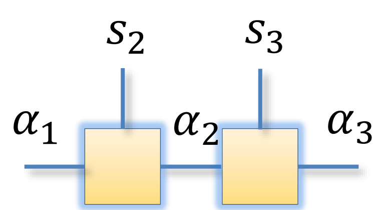
$A^{s_1}_{\alpha_1}$ =



$\alpha_i = 1,2, \ldots, D_i$

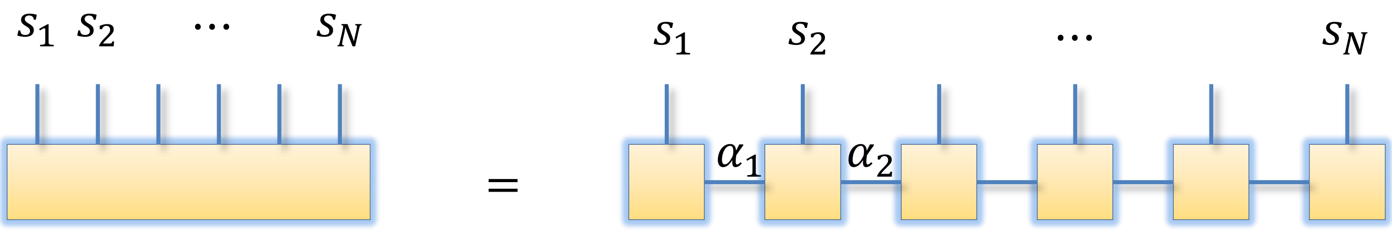
D_i: bond dimension

$\sum_{\alpha_2} A^{s_2}_{\alpha_1 \alpha_2} A^{s_3}_{\alpha_2 \alpha_3}$ =



§ 1. Matrix Product State

Matrix Product State:

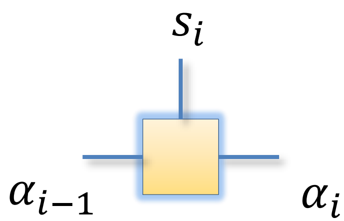


$$\psi(s_1, \dots, s_N) = \sum_{\alpha_1, \alpha_2, \dots, \alpha_{N-1}} A^{s_1}_{\alpha_1} A^{s_2}_{\alpha_1 \alpha_2} \cdots A^{s_{N-1}}_{\alpha_{N-2} \alpha_{N-1}} A^{s_N}_{\alpha_{N-1}}$$

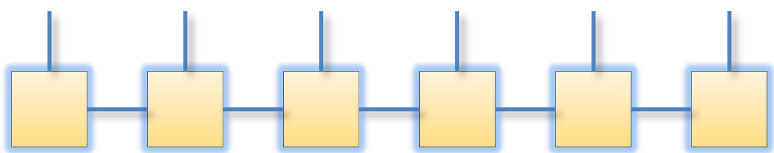
§ 1. Matrix Product State

Advantage 1:

# of parameters scales polynomially with the system size N.



# of parameters $= 2D_{i-1}D_i$

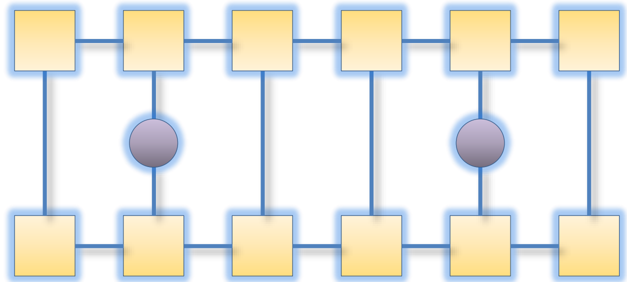


# of parameters $\sim N \cdot 2D_{\max}^2$

§ 1. Matrix Product State

Advantage 2: Physical quantities can be efficiently computed.

$\langle\psi|\sigma_i^z\sigma_j^z|\psi\rangle \quad =$



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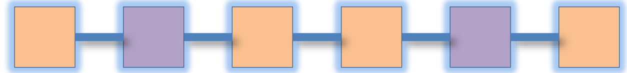
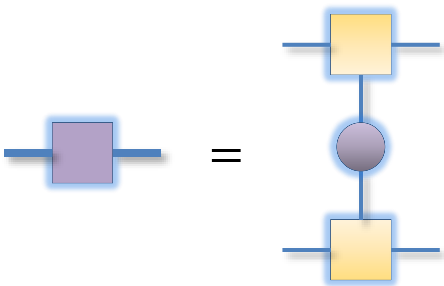
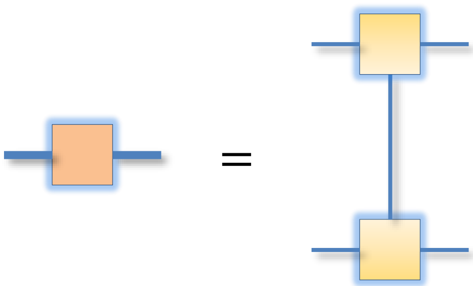


Matrix multiplications!

Transfer matrix:



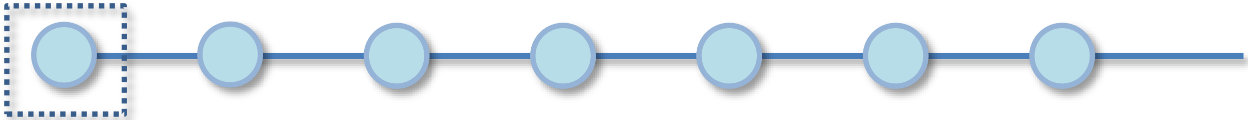


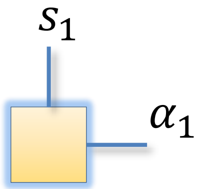
§ 1.1 Real-space renormalization

- Real-space renormalization is a natural way to think of where MPS comes from…

S. Östlund & S. Rommer, Phys. Rev. Lett. 75, 3537 (1995);
S. Rommer & S. Östlund, Phys. Rev. B 55, 2164 (1997).

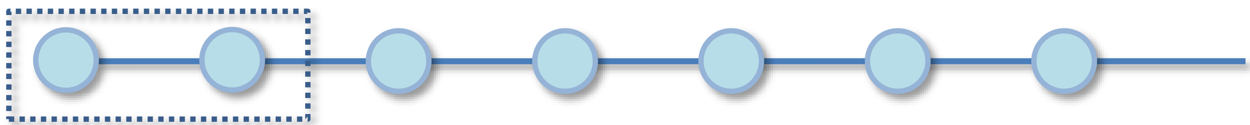
§ 1.1 Real-space renormalization

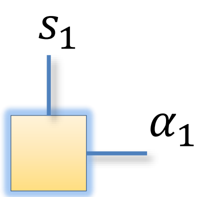




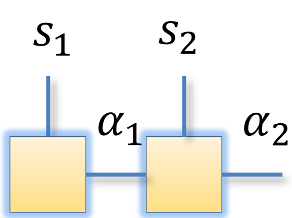
$$|\alpha_1\rangle = \sum_{s_1} A^{s_1}_{\alpha_1} |s_1\rangle$$

§ 1.1 Real-space renormalization



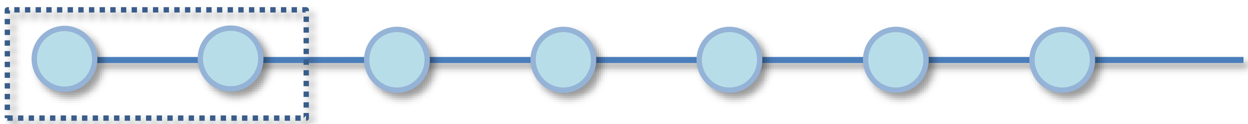


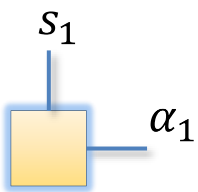
$$|\alpha_1\rangle = \sum_{s_1} A_{\alpha_1}^{s_1} \, |s_1\rangle$$



$$|\alpha_2\rangle = \sum_{\alpha_1, s_2} A_{\alpha_1 s_2}^{\alpha_2} \, |\alpha_1\rangle |s_2\rangle$$

§ 1.1 Real-space renormalization





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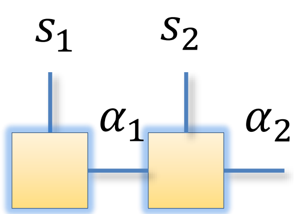

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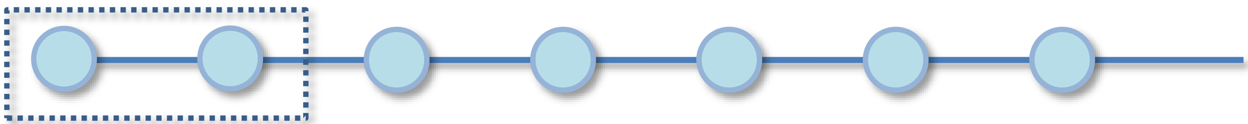

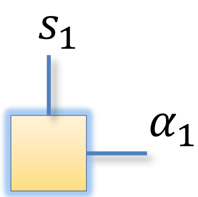

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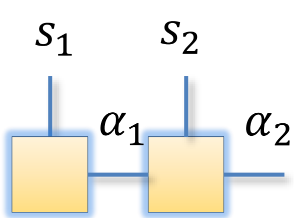

$$|\alpha_2\rangle = \sum_{\alpha_1, s_2} A_{\alpha_1 s_2}^{\alpha_2} |\alpha_1\rangle |s_2\rangle = \sum_{\alpha_1, s_1, s_2} A_{\alpha_1}^{s_1} A_{\alpha_1 \alpha_2}^{s_2} |s_1\rangle |s_2\rangle$$

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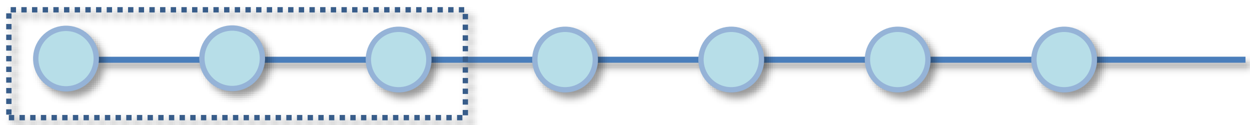
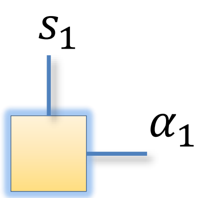
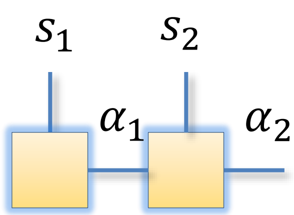




$$|\alpha_1\rangle = \sum_{s_1} A^{s_1}_{\alpha_1} |s_1\rangle$$



$$|\alpha_2\rangle = \sum_{\alpha_1, s_2} A^{\alpha_2}_{\alpha_1 s_2} |\alpha_1\rangle |s_2\rangle = \sum_{\alpha_1, s_1, s_2} A^{s_1}_{\alpha_1} A^{s_2}_{\alpha_1 \alpha_2} |s_1\rangle |s_2\rangle$$

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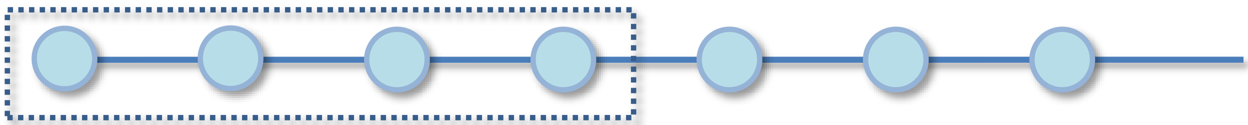

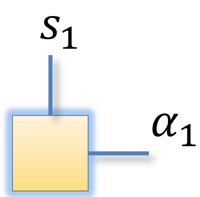

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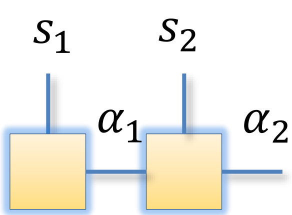
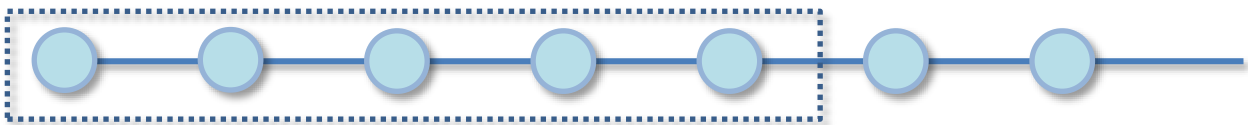
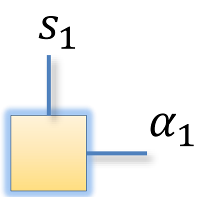


$$|\alpha_2\rangle = \sum_{\alpha_1,s_2} A^{\alpha_2}_{\alpha_1 s_2} |\alpha_1\rangle|s_2\rangle \;=\; \sum_{\alpha_1,s_1,s_2} A^{s_1}_{\alpha_1} A^{s_2}_{\alpha_1 \alpha_2} |s_1\rangle|s_2\rangle$$

§ 1.1 Real-space renormalization





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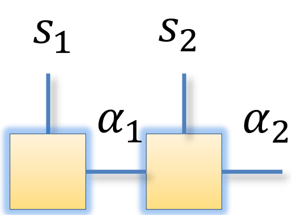
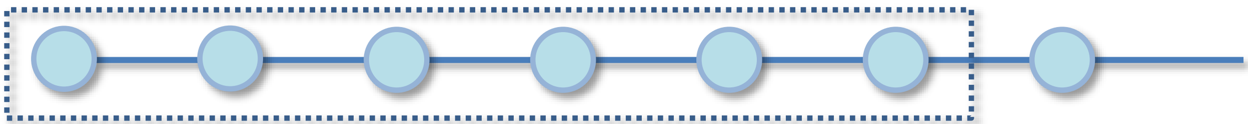
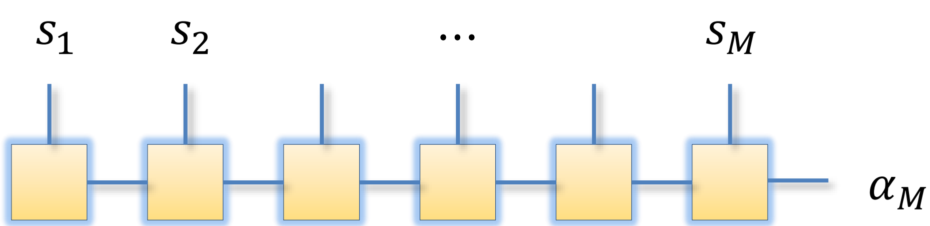


$$|\alpha_2\rangle = \sum_{\alpha_1, s_2} A^{\alpha_2}_{\alpha_1 s_2} |\alpha_1\rangle |s_2\rangle = \sum_{\alpha_1, s_1, s_2} A^{s_1}_{\alpha_1} A^{s_2}_{\alpha_1 \alpha_2} |s_1\rangle |s_2\rangle$$

§ 1.1 Real-space renormalization



After M steps:

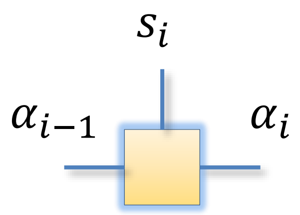


$$|\alpha_M\rangle = \sum_{s_1 \cdots s_M} \sum_{\alpha_1 \ldots \alpha_{M-1}} A^{s_1}_{\alpha_1} A^{s_2}_{\alpha_1 \alpha_2} \cdots A^{s_M}_{\alpha_{M-1}\alpha_M} |s_1\rangle \cdots |s_M\rangle$$

Hilbert space dimension: 2^M ➡ At some point, truncation (keeping fixed # of states) is needed.

§ 1.1 Real-space renormalization

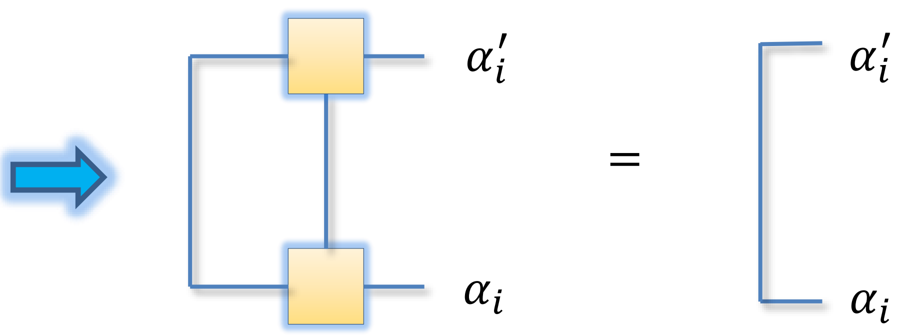
Preserving orthogonality: unitary vs. isometry



$$|\alpha_i\rangle = \sum_{\alpha_{i-1}, s_i} A^{s_i}_{\alpha_{i-1}\alpha_i} |\alpha_{i-1}\rangle |s_i\rangle$$

$$\langle \alpha'_{i-1} | \alpha_{i-1} \rangle = \delta_{\alpha'_{i-1}, \alpha_{i-1}}$$

Requiring $\langle \alpha'_i | \alpha_i \rangle = \delta_{\alpha'_i, \alpha_i}$

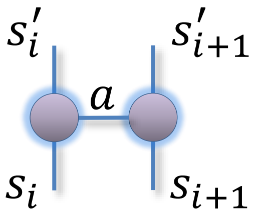


§ 1.2 Wilson's Numerical Renormalization Group

Example: spin-1/2 Heisenberg chain

$$H = \sum_{i=1}^{N-1} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} \qquad \vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$$

$$\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} \quad \rightarrow$$



$$a = x, y, z$$

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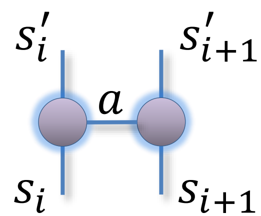

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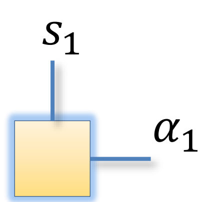
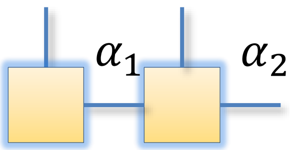
Step 1:



$$|\alpha_1\rangle = \sum_{s_1} A_{\alpha_1}^{s_1} |s_1\rangle \qquad A_{\alpha_1}^{s_1} = \delta_{s_1,\alpha_1}$$

§ 1.2 Wilson's Numerical Renormalization Group

Step 2:



$$|\alpha_2\rangle = \sum_{\alpha_1, s_1, s_2} A^{s_1}_{\alpha_1} A^{s_2}_{\alpha_1 \alpha_2} |s_1\rangle |s_2\rangle$$

Q: How to determine $A^{s_2}_{\alpha_1 \alpha_2}$?

A: Require that $\langle \alpha_2' | \vec{\sigma}_1 \cdot \vec{\sigma}_2 | \alpha_2 \rangle = E_{\alpha_2} \delta_{\alpha_2', \alpha_2}$ and $\langle \alpha_2' | \alpha_2 \rangle = \delta_{\alpha_2', \alpha_2}$.

§ 1.2 Wilson's Numerical Renormalization Group

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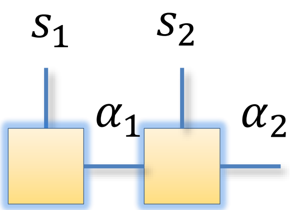

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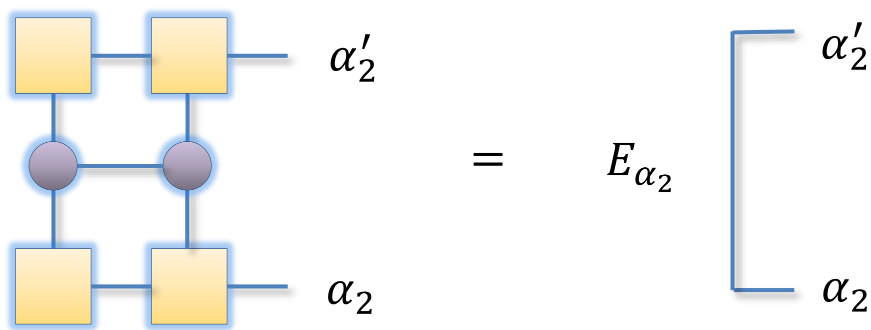

§ 1.2 Wilson's Numerical Renormalization Group

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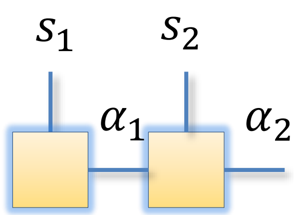

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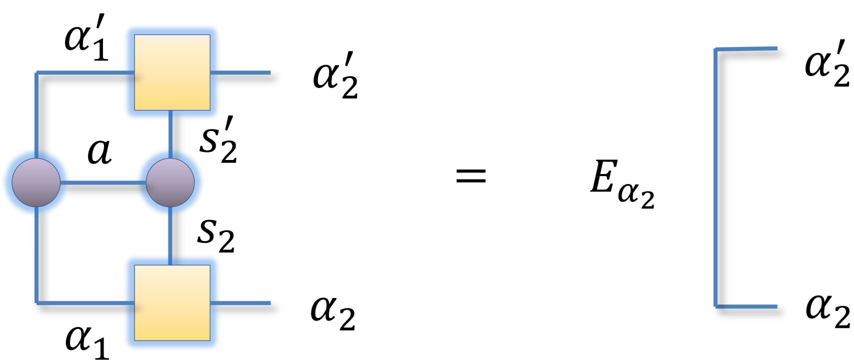

§ 1.2 Wilson's Numerical Renormalization Group

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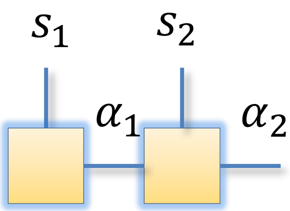

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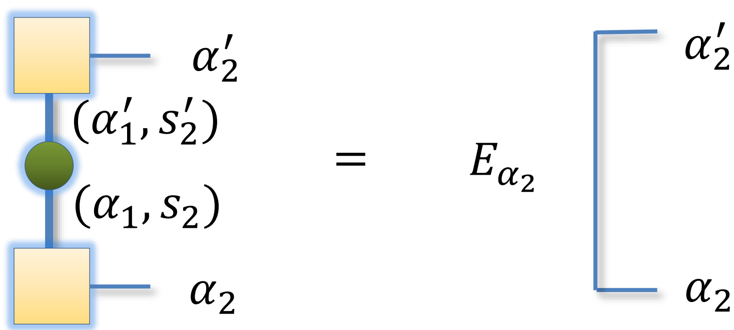
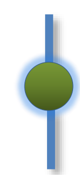
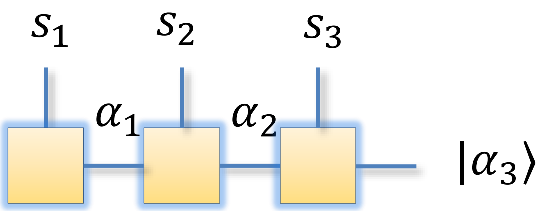


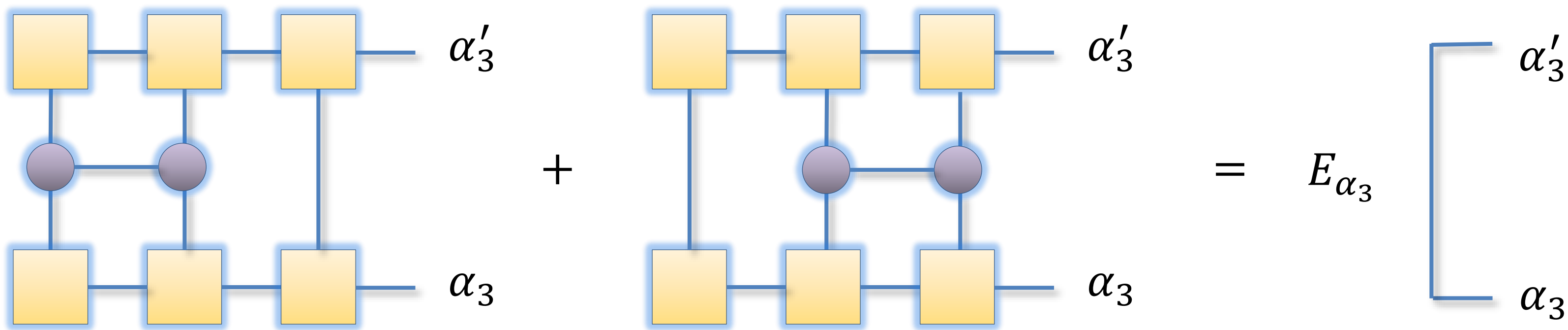
Standard eigenvalue problem! ⇒ $A^{s_2}_{\alpha_1 \alpha_2}$: eigenvectors of 

§ 1.2 Wilson's Numerical Renormalization Group

Step 3:



Diagonalize $\langle\alpha_3'|\vec{\sigma}_1\cdot\vec{\sigma}_2+\vec{\sigma}_2\cdot\vec{\sigma}_3|\alpha_3\rangle$ to determine $A^{s_3}_{\alpha_2\alpha_3}$!



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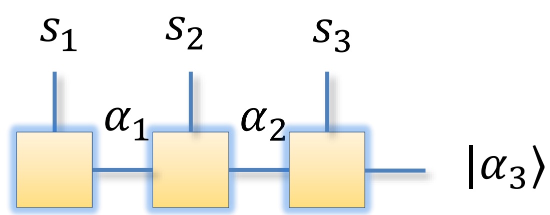

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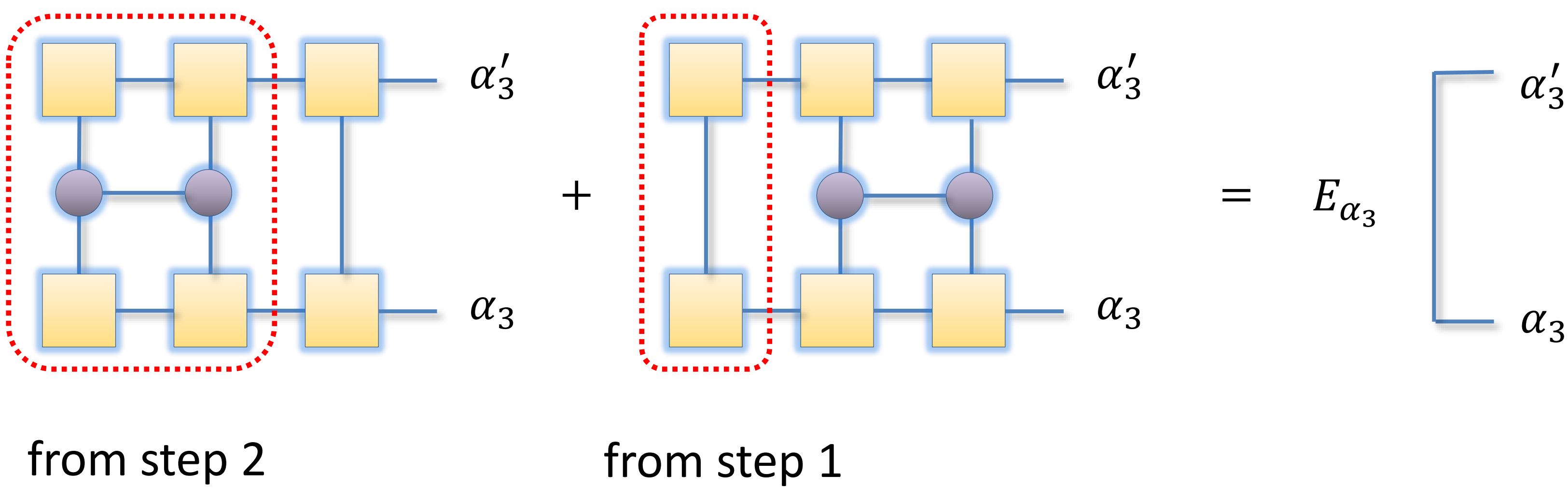

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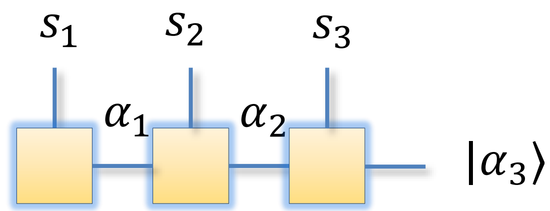
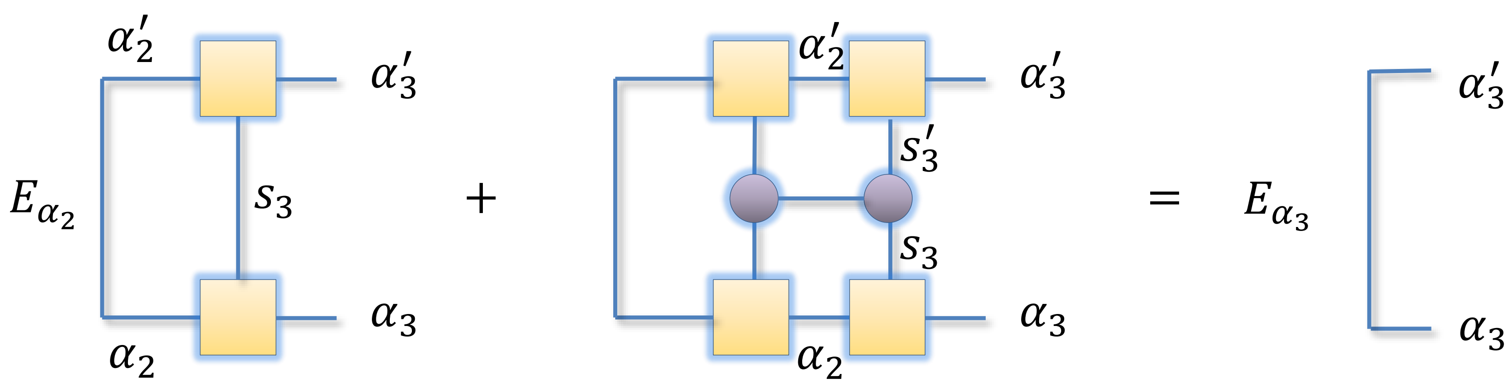
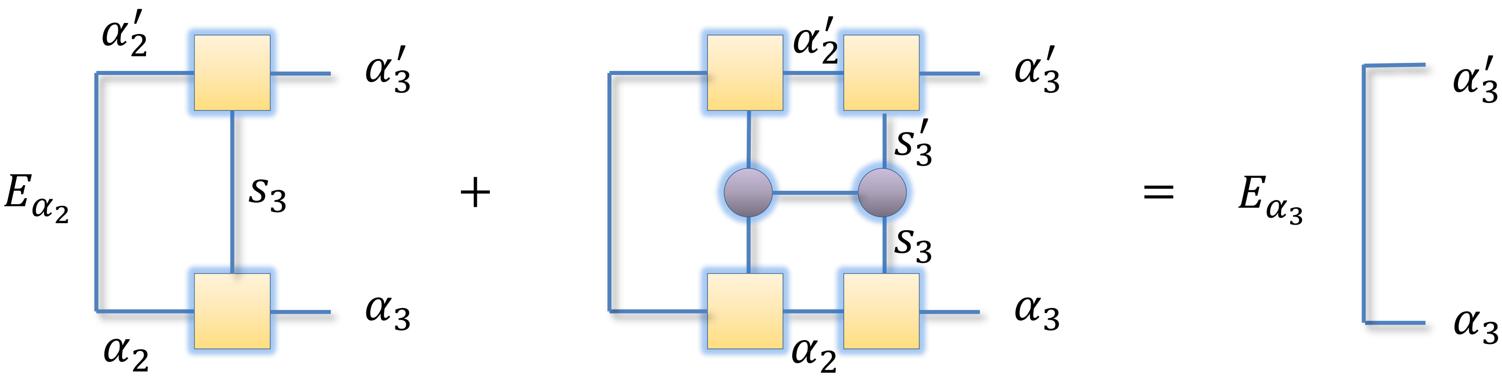
Step 3:

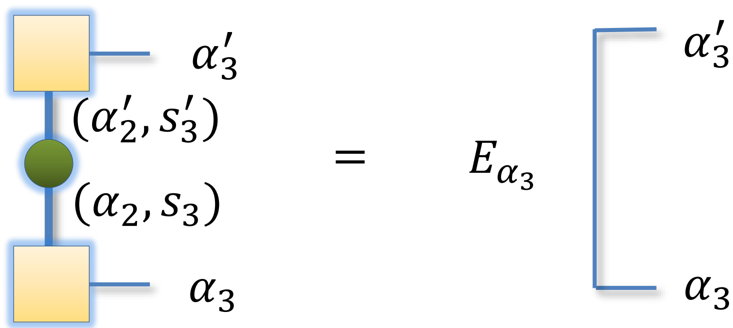


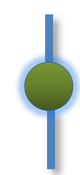
Diagonalize $\langle\alpha_3'|\vec{\sigma}_1\cdot\vec{\sigma}_2+\vec{\sigma}_2\cdot\vec{\sigma}_3|\alpha_3\rangle$ to determine $A^{s_3}_{\alpha_2\alpha_3}$!



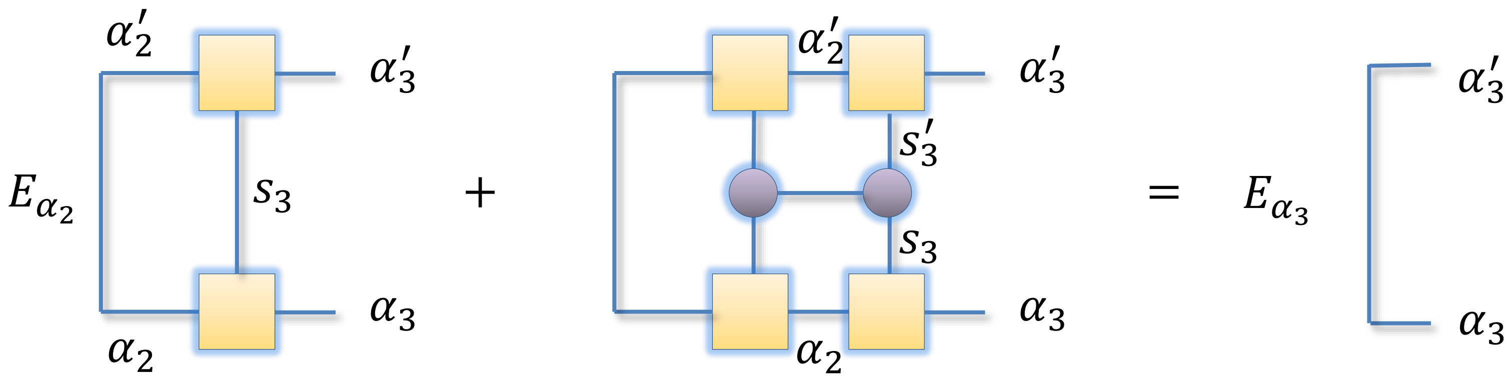
§ 1.2 Wilson's Numerical Renormalization Group

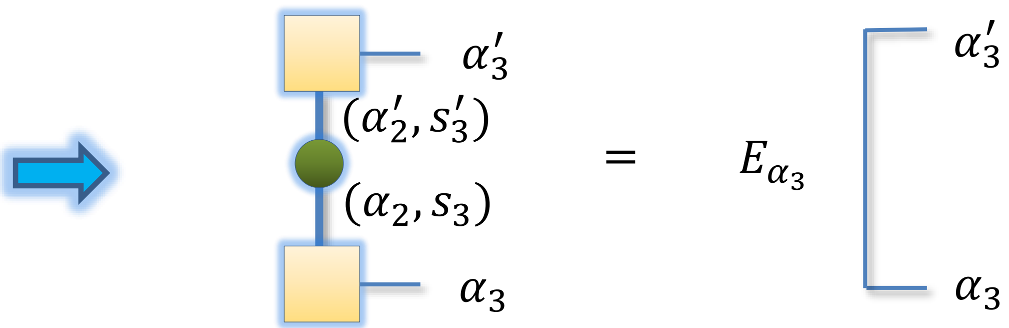




Standard eigenvalue problem! ⇒ $A^{s_3}_{\alpha_2\alpha_3}$: eigenvectors of 

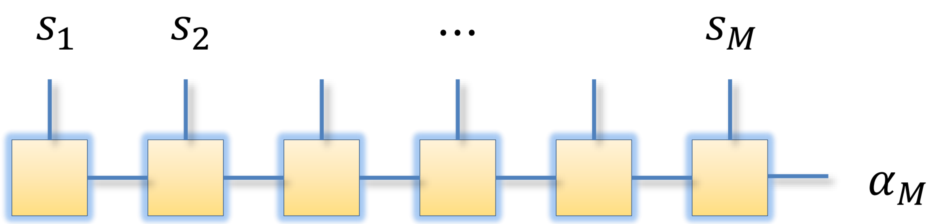
§ 1.2 Wilson's Numerical Renormalization Group





- This iterative diagonalization procedure is easily generalized to longer chains.

§ 1.2 Wilson's Numerical Renormalization Group



Wilson's NRG:

- Iterative diagonalization by keeping at most $D_{\max}$ states (truncation based on the energies of the effective Hamiltonian for the block).

- MPS emerges as the underlying variational ansatz.

- Standard method for Anderson (and Kondo) impurity problems.

K. G. Wilson, Rev. Mod. Phys. 47, 773 (1975);

R. Bulla, T. A. Costi & T. Pruschke, Rev. Mod. Phys. 80, 395 (2008);

A. Weichselbaum, Phys. Rev. B 86, 245124 (2012).