SFB 1143

Tensor Networks (SS2021)

Lecture 4: Entanglement area law + MPS examples

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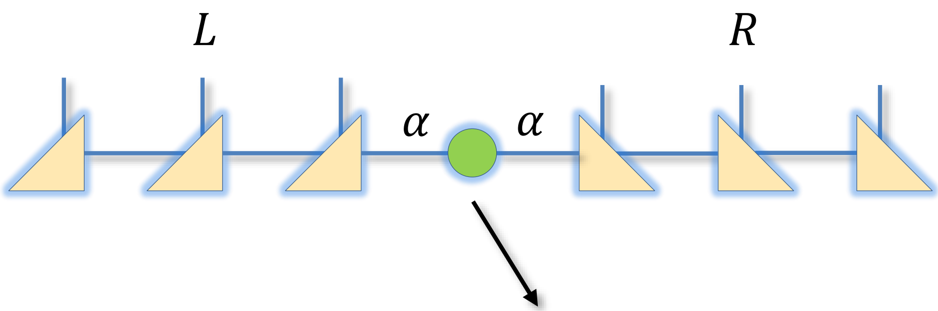
§ 1.3 Canonical forms of MPS

Mixed (bond) canonical form:



Diagonal matrix: $\lambda_\alpha > 0$ and $\sum_{\alpha=1}^{D} \lambda_\alpha^2 = 1$

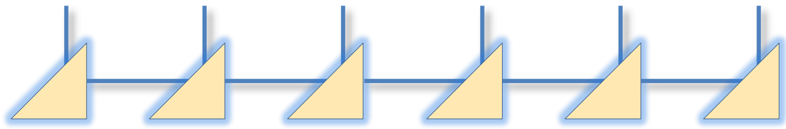
- MPS in mixed canonical form is normalized.

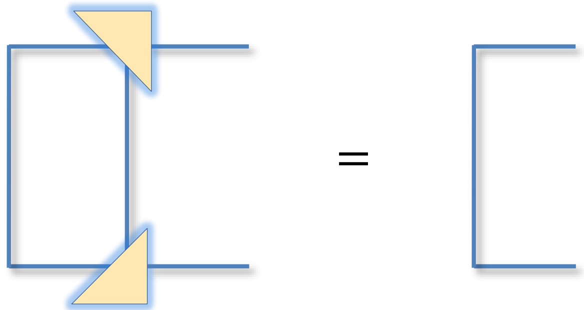
- The mixed canonical form gives the Schmidt decomposition of the bipartite partition:

$$|\psi\rangle = \sum_{\alpha=1}^{D} \lambda_\alpha |\phi_\alpha^L\rangle \otimes |\phi_\alpha^R\rangle$$

§ 1.3 Canonical forms of MPS

Left canonical form:



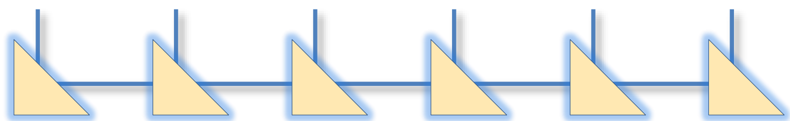


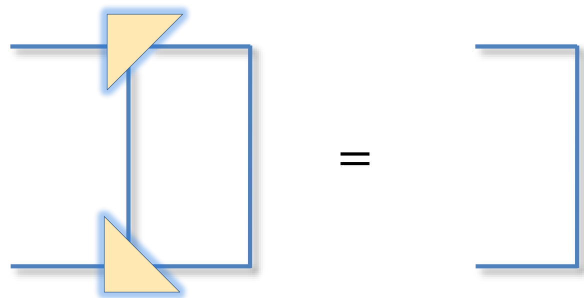
- MPS in left canonical form is normalized.

- The left canonical form is automatically fulfilled if it's generated from the NRG (from left to right).

§ 1.3 Canonical forms of MPS

Right canonical form:

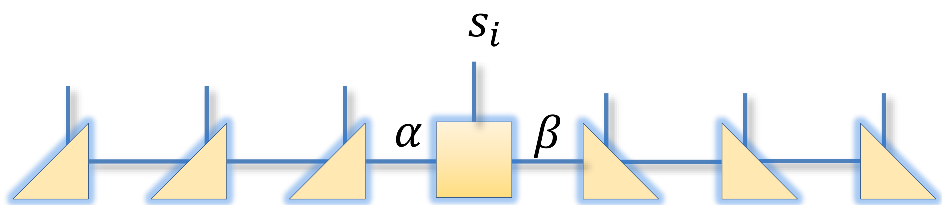




- MPS in right canonical form is normalized.
- The right canonical form is automatically fulfilled if it's generated from the NRG (from right to left).

§ 1.3 Canonical forms of MPS

Site canonical form:



- MPS in site canonical form is normalized if $\sum_{s_i,\alpha\beta} \bar{A}^{s_i}_{\alpha\beta} A^{s_i}_{\alpha\beta} = A^\dagger A = 1$

- This form will be very useful for formulating the DMRG algorithm.

§ 1.3 Canonical forms of MPS

Thin/reduced QR decomposition is also useful:

$$M = QR$$

$m \times n$

Case 1: $m \geq n$

- Q: $m \times n$ isometry
- R: $n \times n$ upper triangular matrix

Case 2: $m < n$ ⇒ $M^T = Q^T R^T$

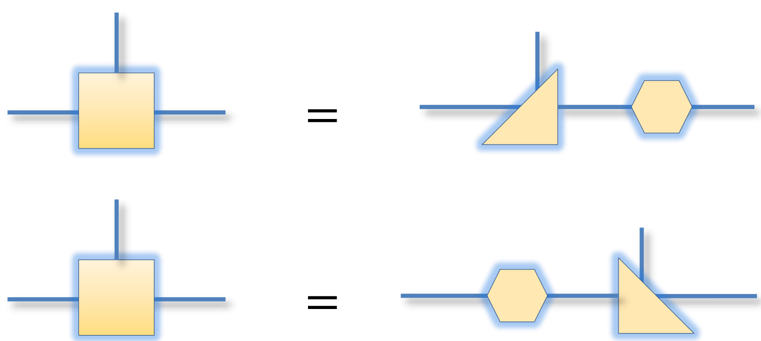
§ 1.3 Canonical forms of MPS

Thin/reduced QR decomposition is also useful:

$$M = QR$$

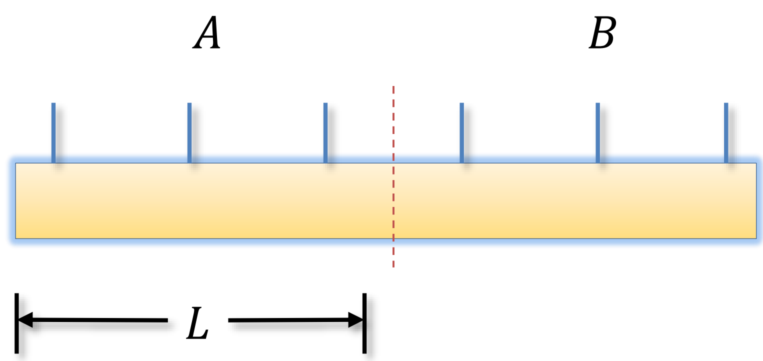
$m \times n$

Use QR decomposition to obtain canonical forms (more efficient):



§ 1.4 Entanglement area law

How large is the entanglement entropy for a random state?



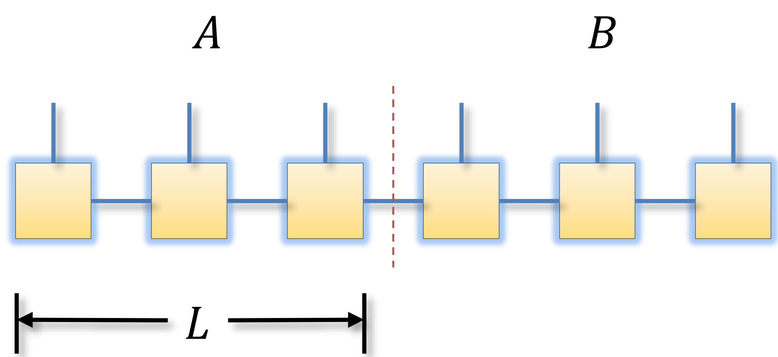
$$|\psi\rangle = \sum_{\alpha=1}^{\chi} \lambda_\alpha |\phi_\alpha^A\rangle \otimes |\phi_\alpha^B\rangle \qquad \chi \sim 2^L$$

Entanglement entropy: $S(L) = -\mathrm{tr}_A(\rho_A \ln \rho_A) \sim \ln \chi \sim L$

volume law (highly entangled!)

§ 1.4 Entanglement area law

Bounded entanglement in MPS:



$$|\psi\rangle = \sum_{\alpha=1}^{\chi} \lambda_\alpha |\phi_\alpha^A\rangle \otimes |\phi_\alpha^B\rangle \qquad \chi \sim D_{\max}$$

Entanglement entropy: $S(L) = -\mathrm{tr}_A(\rho_A \ln \rho_A) \sim \ln \chi \sim \ln(D_{\max})$

area law (low-entanglement states!)

§ 1.4 Entanglement area law

Can we justify the usage of low-entanglement MPS for approximating ground states of realistic systems?

- Rigorous proof of the entanglement area law for gapped ground states of 1d local Hamiltonians

$$S(L) \sim \text{const.}$$

Hastings, J. Stat. Mech. (2007) P08024.

§ 1.4 Entanglement area law

Can we justify the usage of low-entanglement MPS for approximating ground states of realistic systems?

- Logarithmic violation of the area law for gapless ground states described by conformal field theory (CFT)

$$S(L) \sim \begin{cases} \frac{c}{6} \ln L & \text{open boundary condition (OBC)} \\ \frac{c}{3} \ln L & \text{periodic boundary condition (PBC)} \end{cases}$$

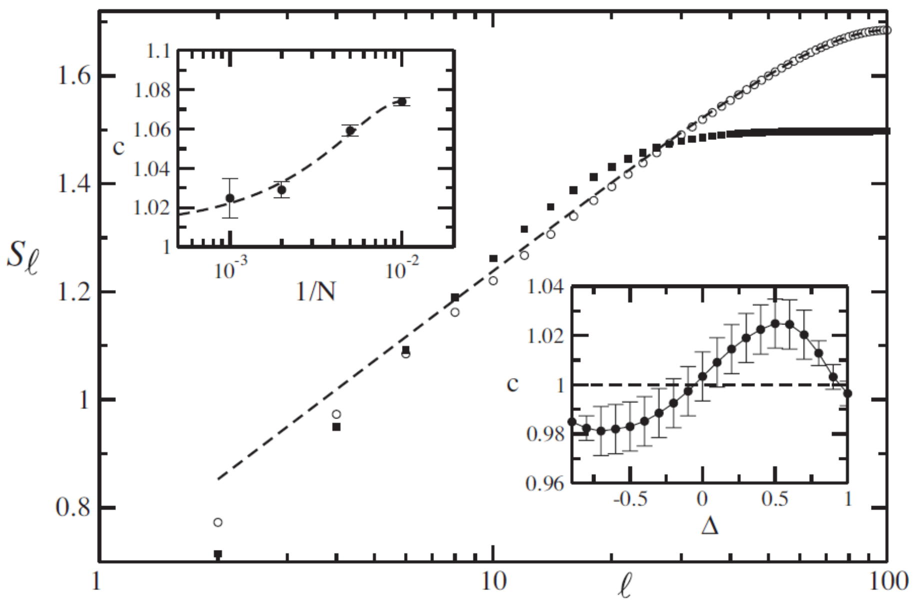
Holzhey, Larsen & Wilczek, Nucl. Phys. B 424, 443 (1994);

Vidal, Latorre, Rico & Kitaev, Phys. Rev. Lett. 90, 227902 (2003);

Calabrese & Cardy, J. Stat. Mech. (2004) P06002.

§ 1.4 Entanglement area law

Example: spin-1/2 XXZ model $H = \sum_{i=1}^{N-1} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z)$



$\Delta = 0$ (gapless, CFT)

$\Delta = 1.8$ (gapped)

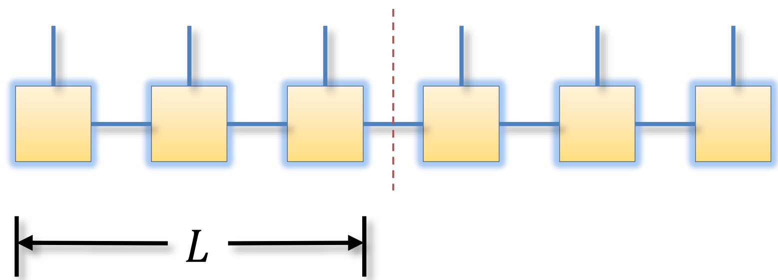
$N = 200$

$S_l = -\mathrm{tr}_l(\rho_l \log_2 \rho_l)$ used!

De Chiara, Montangero, Calabrese & Fazio, J. Stat. Mech. (2006) P03001.

§ 1.4 Entanglement area law

Required bound dimension for MPS:



$$D_{\max} \sim e^{S(L)} \sim \begin{cases} \text{const.} & \text{gapped} \\ L^{c/6} & \text{gapless (CFT with OBC)} \end{cases}$$

§ 1.4 Entanglement area law

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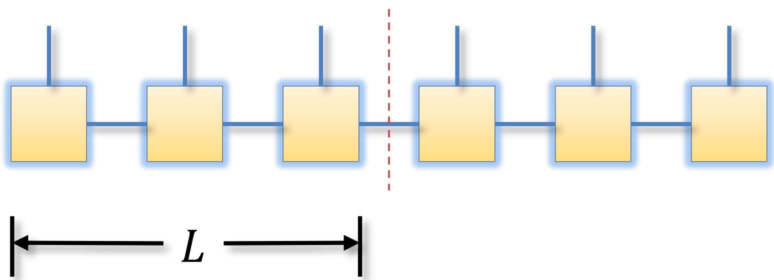


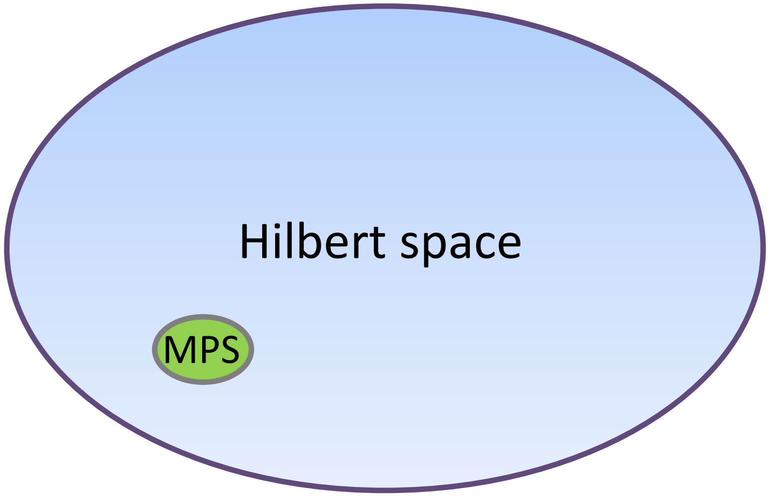
$$D_{\max} \sim e^{S(L)} \sim \begin{cases} \text{const.} & \text{gapped} \\ L^{c/6} & \text{gapless (CFT with OBC)} \end{cases}$$

States with volume-law entanglement? $D_{\max} \sim e^{S(L)} \sim e^{L}$

§ 1.4 Entanglement area law

- MPS are low-entanglement states satisfying the entanglement area law.

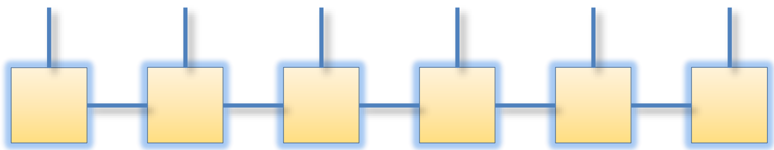
- Local interactions generate small amount of entanglement (area law + logarithmic violations).



§ 1.5 MPS examples

- Many interesting states have an exact MPS form (with small bond dimension).

- To get some intuitions, we could write the MPS as follows:



$$|\psi\rangle = \sum_{s_1 \dots s_N} \sum_{\alpha_1,\dots,\alpha_{N-1}} A^{s_1}_{\alpha_1} A^{s_2}_{\alpha_1\alpha_2} \cdots A^{s_{N-1}}_{\alpha_{N-2}\alpha_{N-1}} A^{s_N}_{\alpha_{N-1}} |s_1 \dots s_N\rangle$$

$$= g_1 g_2 \cdots g_N$$

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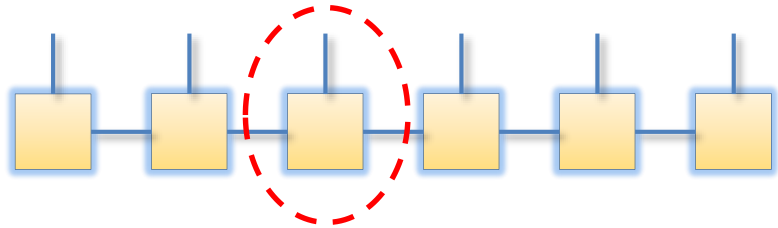

$$|\psi\rangle = \sum_{s_1 \ldots s_N} \sum_{\alpha_1, \ldots, \alpha_{N-1}} A^{s_1}_{\alpha_1} A^{s_2}_{\alpha_1 \alpha_2} \cdots A^{s_{N-1}}_{\alpha_{N-2}\alpha_{N-1}} A^{s_N}_{\alpha_{N-1}} |s_1 \ldots s_N\rangle$$

$$= g_1 g_2 \cdots g_N$$

$$(g_i)_{\alpha_{i-1},\alpha_i} = \sum_{s_i} A^{s_i}_{\alpha_{i-1},\alpha_i} |s_i\rangle$$

§ 1.5 MPS examples

Example 1: GHZ state

$$|\psi_{\mathrm{GHZ}}\rangle = |0, \dots, 0\rangle + |1, \dots, 1\rangle$$

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$$|\psi_{\text{GHZ}}\rangle = |0, \dots, 0\rangle + |1, \dots, 1\rangle$$

$$|\psi_{\text{GHZ}}\rangle = \begin{pmatrix} |0\rangle_1 & |1\rangle_1 \end{pmatrix} \begin{pmatrix} |0\rangle_2 & 0 \\ 0 & |1\rangle_2 \end{pmatrix} \cdots \begin{pmatrix} |0\rangle_{N-1} & 0 \\ 0 & |1\rangle_{N-1} \end{pmatrix} \begin{pmatrix} |0\rangle_N \\ |1\rangle_N \end{pmatrix}$$

$$\Rightarrow \quad A^{s_i=0} = \begin{cases} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \qquad A^{s_i=1} = \begin{cases} \begin{pmatrix} 0 & 1 \end{pmatrix} & i = 1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & i = 2, \dots, N-1 \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & i = N \end{cases}$$

§ 1.5 MPS examples

Example 2: 1d topological superconductor (Kitaev's Majorana chain)

$$|\psi_{\text{even}}\rangle = (1 + a_1^\dagger a_2^\dagger + a_1^\dagger a_3^\dagger + \cdots + a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger + \cdots)|0\rangle$$

$$|\psi_{\text{odd}}\rangle = (a_1^\dagger + a_2^\dagger + \cdots + a_1^\dagger a_2^\dagger a_3^\dagger + \cdots)|0\rangle$$

§ 1.5 MPS examples

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$$= |0,0,\ldots,0,0\rangle + |1,1,\ldots,0,0\rangle + \cdots + |1,1,1,1,\ldots,0,0\rangle + \cdots$$

Equal weight superposition of all possible states with even fermion parity!

$$|\psi_{\mathrm{even}}\rangle = (|0\rangle_1 \quad |1\rangle_1)\begin{pmatrix}|0\rangle_2 & |1\rangle_2\\ |1\rangle_2 & |0\rangle_2\end{pmatrix}\cdots\begin{pmatrix}|0\rangle_{N-1} & |1\rangle_{N-1}\\ |1\rangle_{N-1} & |0\rangle_{N-1}\end{pmatrix}\begin{pmatrix}|0\rangle_N\\ |1\rangle_N\end{pmatrix}$$

§ 1.5 MPS examples

Example 2: 1d topological superconductor (Kitaev's Majorana chain)

$$|\psi_{\text{odd}}\rangle = (a_1^\dagger + a_2^\dagger + \cdots + a_1^\dagger a_2^\dagger a_3^\dagger + \cdots)|0\rangle$$

$$= |1,0,\dots,0,0\rangle + |0,1,\dots,0,0\rangle + \cdots + |1,1,1,0,\dots,0,0\rangle + \cdots$$

Equal weight superposition of all possible states with odd fermion parity!

$$|\psi_{\text{odd}}\rangle = (|1\rangle_1 \quad |0\rangle_1) \begin{pmatrix} |0\rangle_2 & |1\rangle_2 \\ |1\rangle_2 & |0\rangle_2 \end{pmatrix} \cdots \begin{pmatrix} |0\rangle_{N-1} & |1\rangle_{N-1} \\ |1\rangle_{N-1} & |0\rangle_{N-1} \end{pmatrix} \begin{pmatrix} |0\rangle_N \\ |1\rangle_N \end{pmatrix}$$