TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

SFB 1143

Tensor Networks (SS2021)

Lecture 5: AKLT model

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§ 1.6 AKLT model

Spin-S antiferromagnetic Heisenberg chain:

$$H = \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} \qquad \vec{S}_i^2 = S(S+1)$$

Haldane’s conjecture (Nobel Prize 2016):

- $S =$ integer: gapped
- $S =$ half integer: gapless

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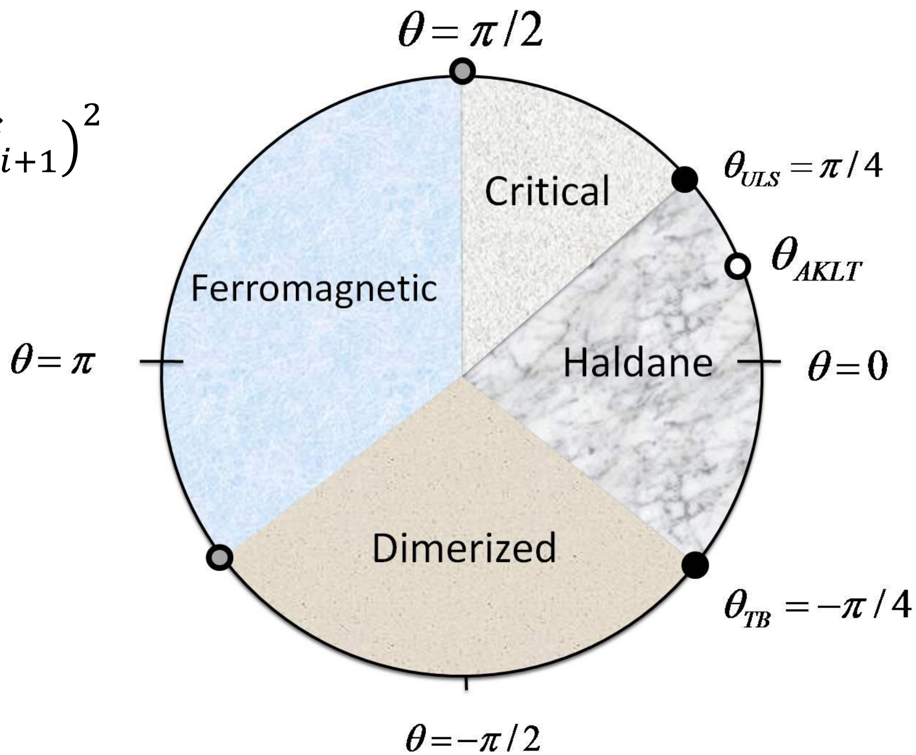
Spin-1 bilinear-biquadratic chain:

$$H = \sum_{i=1}^{N} \cos\theta\,(\vec{S}_i \cdot \vec{S}_{i+1}) + \sin\theta\,\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2$$



Haldane phase:

- Unique gapped ground state for periodic chain, 4-fold degeneracy for open chain

- Exponentially decaying spin correlations

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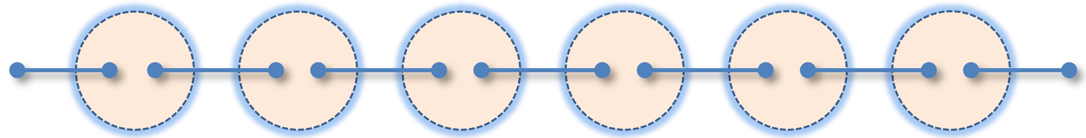

Affleck, Kennedy, Lieb & Tasaki, Phys. Rev. Lett. 59, 799 (1987).

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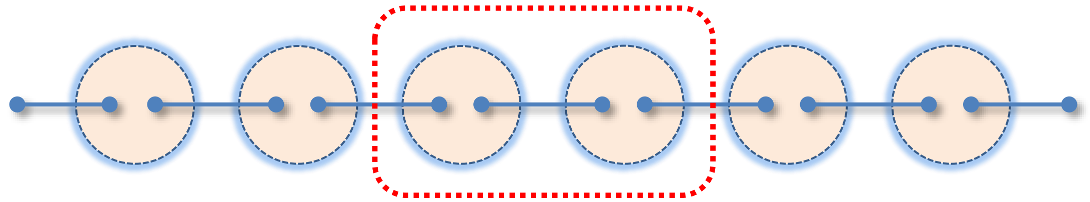

Physical: $1 \otimes 1 = 0 \oplus 1 \oplus 2$

Virtual: $\frac{1}{2} \otimes \underline{\frac{1}{2} \otimes \frac{1}{2}} \otimes \frac{1}{2} = 0 \oplus 1$

$\downarrow$

0

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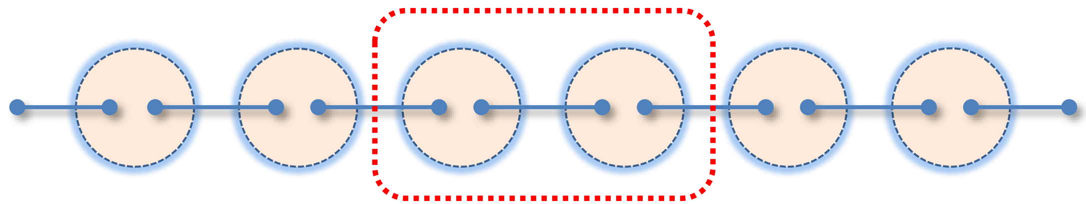

Physical: $1 \otimes 1 = 0 \oplus 1 \oplus 2$

Virtual: $\frac{1}{2} \otimes \underbrace{\frac{1}{2} \otimes \frac{1}{2}}_{0} \otimes \frac{1}{2} = 0 \oplus 1$

"2": Absence of total spin $S_t = 2$ states for every two neighboring sites

$$H_{\text{AKLT}} = \sum_i P_{i,i+1}^{S_t=2}$$

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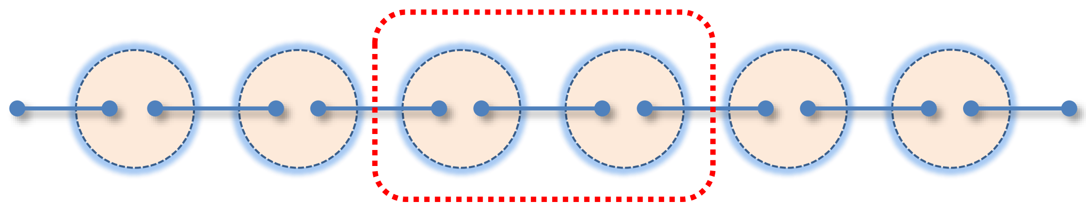

Physical: $1 \otimes 1 = 0 \oplus 1 \oplus 2$

Deriving the explicit form of two-site projectors $P_{i,i+1}^{S_t}$:

$$1 = P_{i,i+1}^{S_t=0} + P_{i,i+1}^{S_t=1} + P_{i,i+1}^{S_t=2}$$

$$\vec{S}_i \cdot \vec{S}_{i+1} = \frac{1}{2}\left[\left(\vec{S}_i + \vec{S}_{i+1}\right)^2 - \vec{S}_i^2 - \vec{S}_{i+1}^2\right] = \sum_{S_t=0,1,2} \frac{1}{2} S_t(S_t+1) P_{i,i+1}^{S_t} - 1(1+1)$$

$$= -2P_{i,i+1}^{S_t=0} - P_{i,i+1}^{S_t=1} + P_{i,i+1}^{S_t=2}$$

$$\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 = (-2)^2 P_{i,i+1}^{S_t=0} + (-1)^2 P_{i,i+1}^{S_t=1} + P_{i,i+1}^{S_t=2} = 4P_{i,i+1}^{S_t=0} + P_{i,i+1}^{S_t=1} + P_{i,i+1}^{S_t=2}$$

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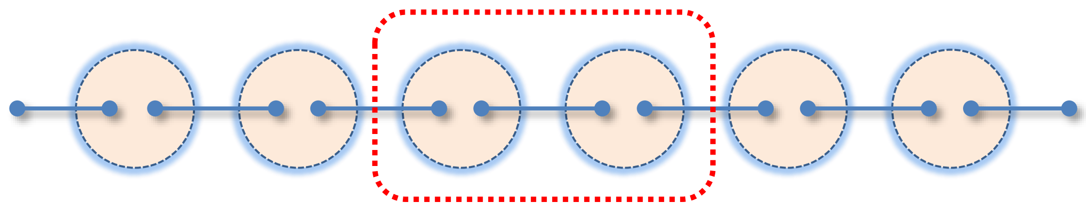

Physical: $1 \otimes 1 = 0 \oplus 1 \oplus 2$

Deriving the explicit form of two-site projectors $P_{i,i+1}^{S_t}$:

$$P_{i,i+1}^{S_t=0} = \frac{1}{3}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 - \frac{1}{3}$$

$$P_{i,i+1}^{S_t=1} = -\frac{1}{2}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right) - \frac{1}{2}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 + 1$$

$$P_{i,i+1}^{S_t=2} = \frac{1}{2}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right) + \frac{1}{6}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 + \frac{1}{3}$$

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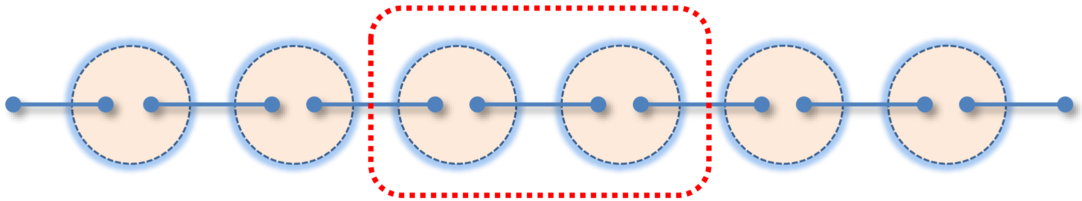

Physical: $1 \otimes 1 = 0 \oplus 1 \oplus 2$

Virtual: $\frac{1}{2} \otimes \underline{\frac{1}{2} \otimes \frac{1}{2}} \otimes \frac{1}{2} = 0 \oplus 1$

0

"2": Absence of total spin $S_t = 2$ states for every two neighboring sites

$$H_{\mathrm{AKLT}} = \sum_i P_{i,i+1}^{S_t=2} = \sum_i \frac{1}{2}(\vec{S}_i \cdot \vec{S}_{i+1}) + \frac{1}{6}(\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{1}{3}$$

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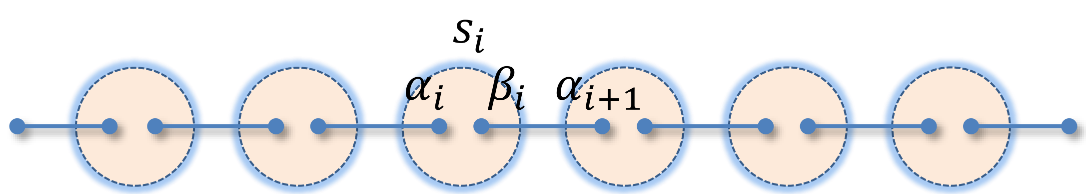

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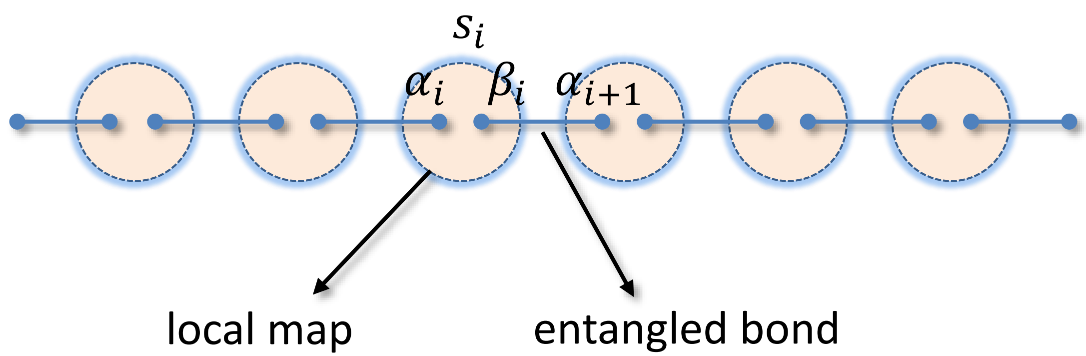

$$P_i = \sum_{s_i=1}^{d} \sum_{\alpha_i,\beta_i=1}^{D} B^{s_i}_{\alpha_i\beta_i} \, |s_i\rangle\langle\alpha_i,\beta_i| \qquad |I\rangle_{i,i+1} = \sum_{\beta_i,\alpha_{i+1}=1}^{D} R_{\beta_i,\alpha_{i+1}} \, |\beta_i,\alpha_{i+1}\rangle$$

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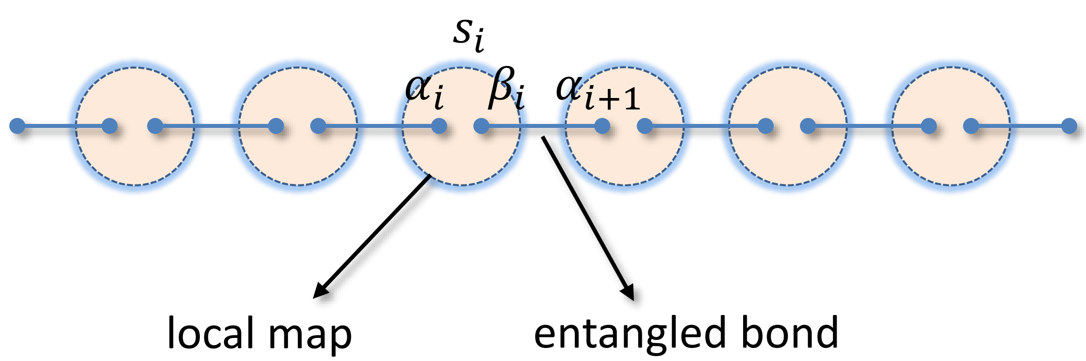

$$P_i = \sum_{s_i=1}^{d} \sum_{\alpha_i,\beta_i=1}^{D} B_{\alpha_i\beta_i}^{s_i} |s_i\rangle\langle\alpha_i,\beta_i| \qquad |I\rangle_{i,i+1} = \sum_{\beta_i,\alpha_{i+1}=1}^{D} R_{\beta_i,\alpha_{i+1}} |\beta_i,\alpha_{i+1}\rangle$$

$$|\psi\rangle = \left(\bigotimes_{i=1}^{N} P_i\right)\left(\bigotimes_{j=1}^{N} |I\rangle_{j,j+1}\right)$$

Periodic boundary: $1 \leftrightarrow N+1$

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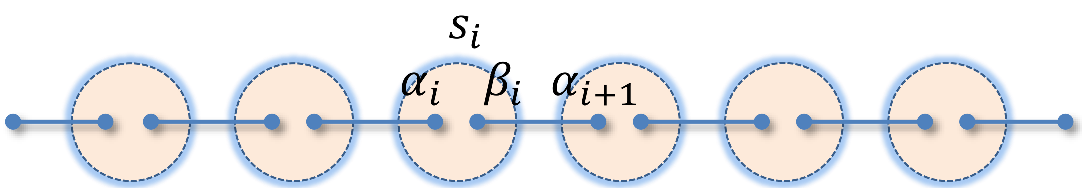

$$|\psi\rangle = \left(\bigotimes_{i=1}^{N} P_i\right)\left(\bigotimes_{j=1}^{N} |I\rangle_{j,j+1}\right)$$

$$= \sum_{s_1 \dots s_N=1}^{d} \sum_{\alpha_1,\beta_1 \dots \alpha_N,\beta_N=1}^{D} \left(B^{s_1}_{\alpha_1\beta_1} \cdots B^{s_N}_{\alpha_N\beta_N}\right) |s_1 \dots s_N\rangle\langle\alpha_1,\beta_1 \dots \alpha_N,\beta_N|$$

$$\times \sum_{\alpha'_1,\beta'_1 \dots \alpha'_N,\beta'_N=1}^{D} \left(R_{\beta'_1,\alpha'_2} \cdots R_{\beta'_N,\alpha'_1}\right) |\alpha'_1,\beta'_1 \dots \alpha'_N,\beta'_N\rangle$$

$$= \sum_{s_1 \dots s_N=1}^{d} \sum_{\alpha_1,\beta_1 \dots \alpha_N,\beta_N=1}^{D} \left(B^{s_1}_{\alpha_1\beta_1} R_{\beta_1,\alpha_2} \cdots B^{s_N}_{\alpha_N\beta_N} R_{\beta_N,\alpha_1}\right) |s_1 \dots s_N\rangle$$

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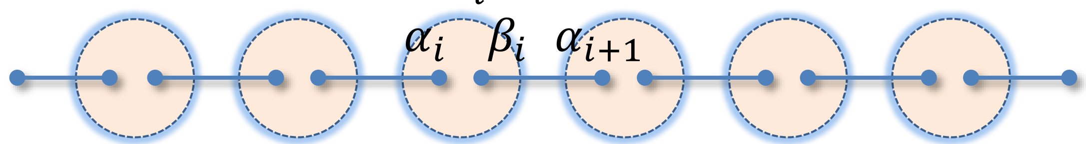
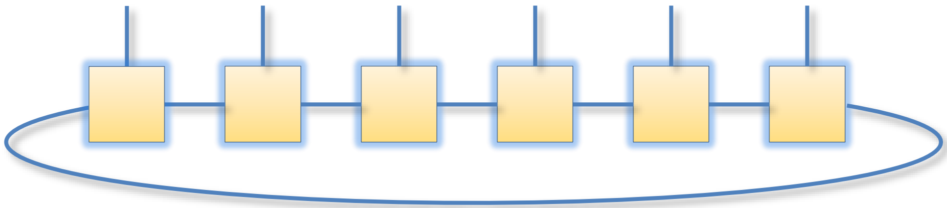


$$P_i = \sum_{s_i=1}^{d} \sum_{\alpha_i,\beta_i=1}^{D} B^{s_i}_{\alpha_i\beta_i} |s_i\rangle\langle\alpha_i,\beta_i| \qquad |I\rangle_{i,i+1} = \sum_{\beta_i,\alpha_{i+1}=1}^{D} R_{\beta_i,\alpha_{i+1}} |\beta_i,\alpha_{i+1}\rangle$$

$$|\psi_{\mathrm{PBC}}\rangle = \sum_{s_1,s_2,\ldots,s_N=1}^{d} \mathrm{Tr}\,(A^{s_1}A^{s_2}\cdots A^{s_N})|s_1,s_2,\ldots,s_N\rangle \qquad A^{s_i} = B^{s_i}R$$



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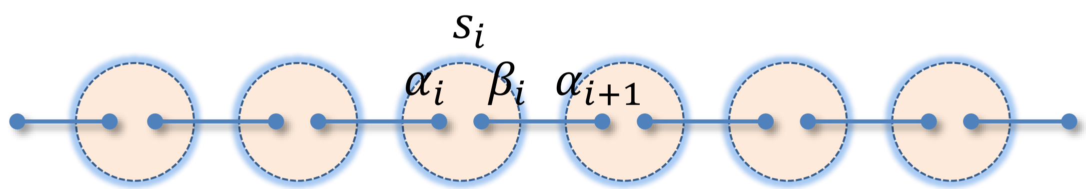
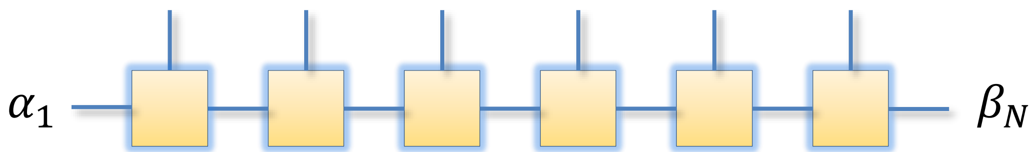


$$P_i = \sum_{s_i=1}^{d} \sum_{\alpha_i,\beta_i=1}^{D} B^{s_i}_{\alpha_i\beta_i} |s_i\rangle\langle\alpha_i,\beta_i| \qquad |I\rangle_{i,i+1} = \sum_{\beta_i,\alpha_{i+1}=1}^{D} R_{\beta_i,\alpha_{i+1}} |\beta_i,\alpha_{i+1}\rangle$$

$$|\psi_{\mathrm{OBC}}(\alpha_1\beta_N)\rangle = \sum_{s_1,s_2,\ldots,s_N=1}^{d} \left(A^{s_1}A^{s_2}\cdots\tilde{A}^{s_N}\right)_{\alpha_1\beta_N} |s_1,s_2,\ldots,s_N\rangle \qquad \tilde{A}^{s_N} = B^{s_N}$$



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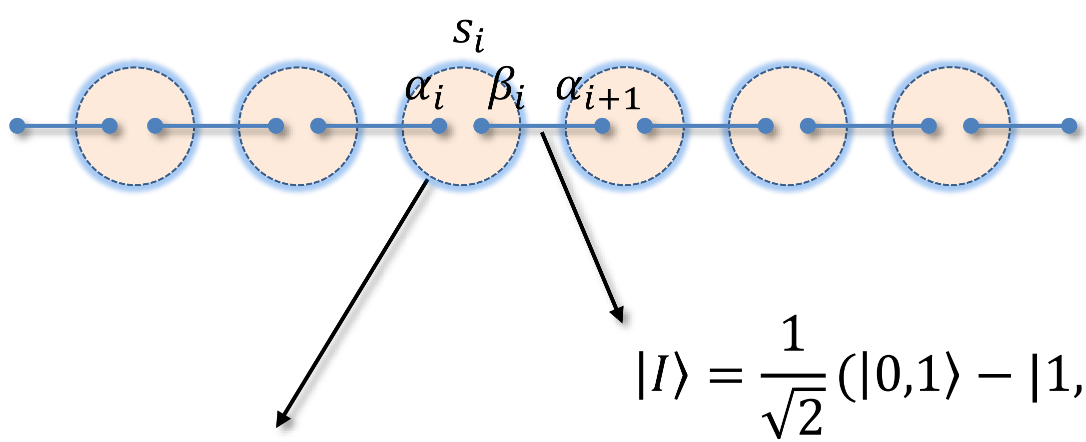

$$|I\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)$$

$$P = |1\rangle\langle 0,0| + \frac{1}{\sqrt{2}}|0\rangle(\langle 0,1| + \langle 1,0|) + |-1\rangle\langle 1,1|$$

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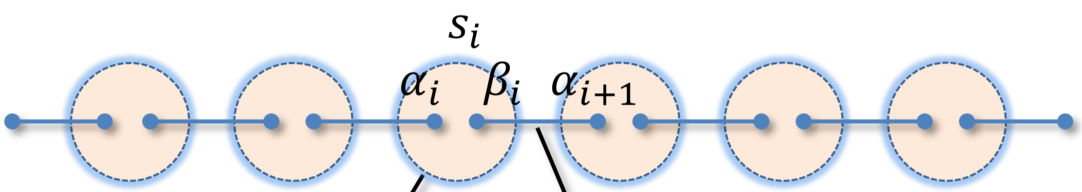

$$|I\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)$$

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$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$B^{s=1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B^{s=0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad B^{s=-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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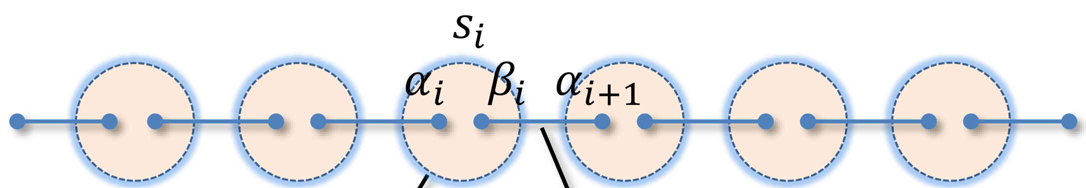

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$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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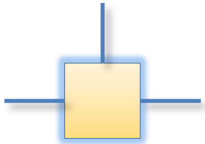

$$A^{s=1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad A^{s=0} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^{s=-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

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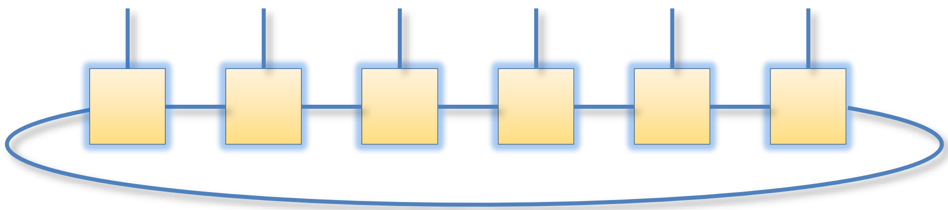

$$|\psi_{\mathrm{AKLT}}\rangle = \sum_{s_1, s_2, \ldots, s_N} \mathrm{Tr}\,(A^{s_1} A^{s_2} \cdots A^{s_N}) |s_1, s_2, \ldots, s_N\rangle$$

$$= \mathrm{Tr}(g_1 g_2 \cdots g_N)$$

$$g_i = \sum_{s_i = -1,0,1} A^{s_i} |s_i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -|0\rangle_i & \sqrt{2}|1\rangle_i \\ -\sqrt{2}|-1\rangle_i & |0\rangle_i \end{pmatrix}$$

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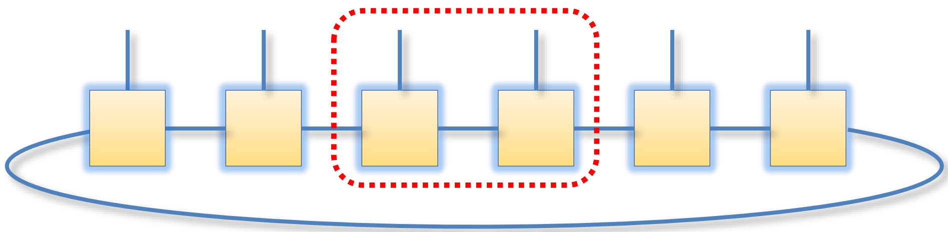

$$g_i g_{i+1} = \frac{1}{2}\begin{pmatrix} -|0\rangle_i & \sqrt{2}|1\rangle_i \\ -\sqrt{2}|-1\rangle_i & |0\rangle_i \end{pmatrix}\begin{pmatrix} -|0\rangle_{i+1} & \sqrt{2}|1\rangle_{i+1} \\ -\sqrt{2}|-1\rangle_{i+1} & |0\rangle_{i+1} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} |0\rangle_i|0\rangle_{i+1} - 2|1\rangle_i|-1\rangle_{i+1} & \sqrt{2}(|1\rangle_i|0\rangle_{i+1} - |0\rangle_i|1\rangle_{i+1}) \\ \sqrt{2}(|-1\rangle_i|0\rangle_{i+1} - |0\rangle_i|-1\rangle_{i+1}) & |0\rangle_i|0\rangle_{i+1} - 2|-1\rangle_i|1\rangle_{i+1} \end{pmatrix}$$

The (five) total spin-2 states are indeed absent in the AKLT state.

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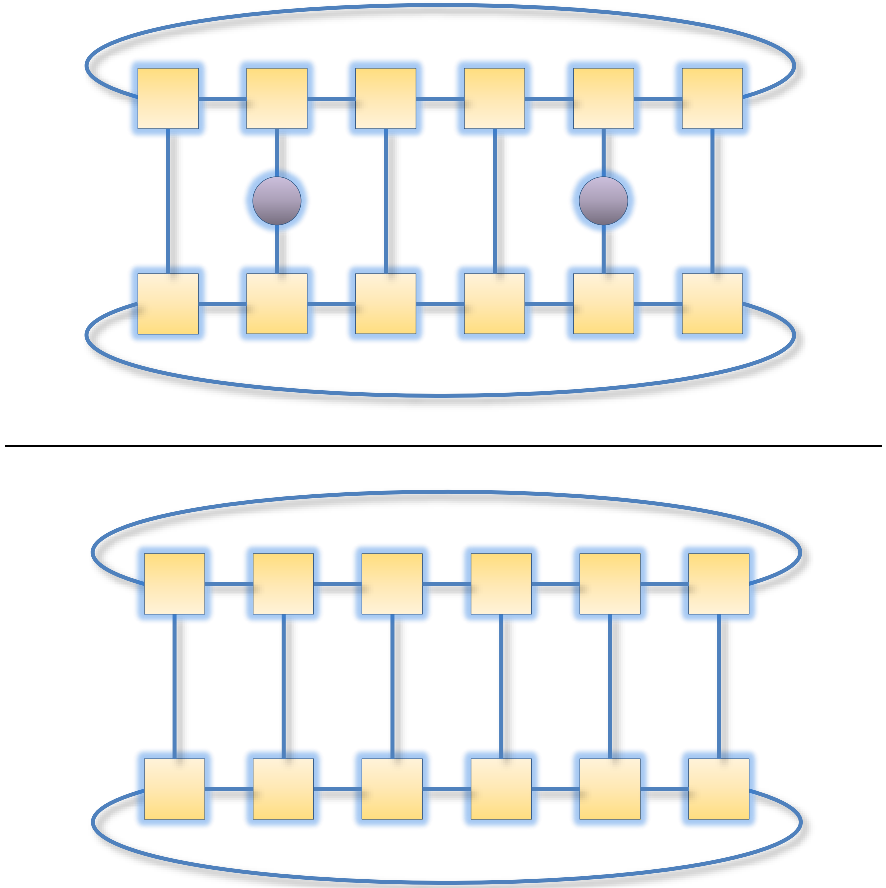
Correlation function:

$$\langle O_i O_j \rangle = \frac{\langle \psi | O_i O_j | \psi \rangle}{\langle \psi | \psi \rangle} =$$



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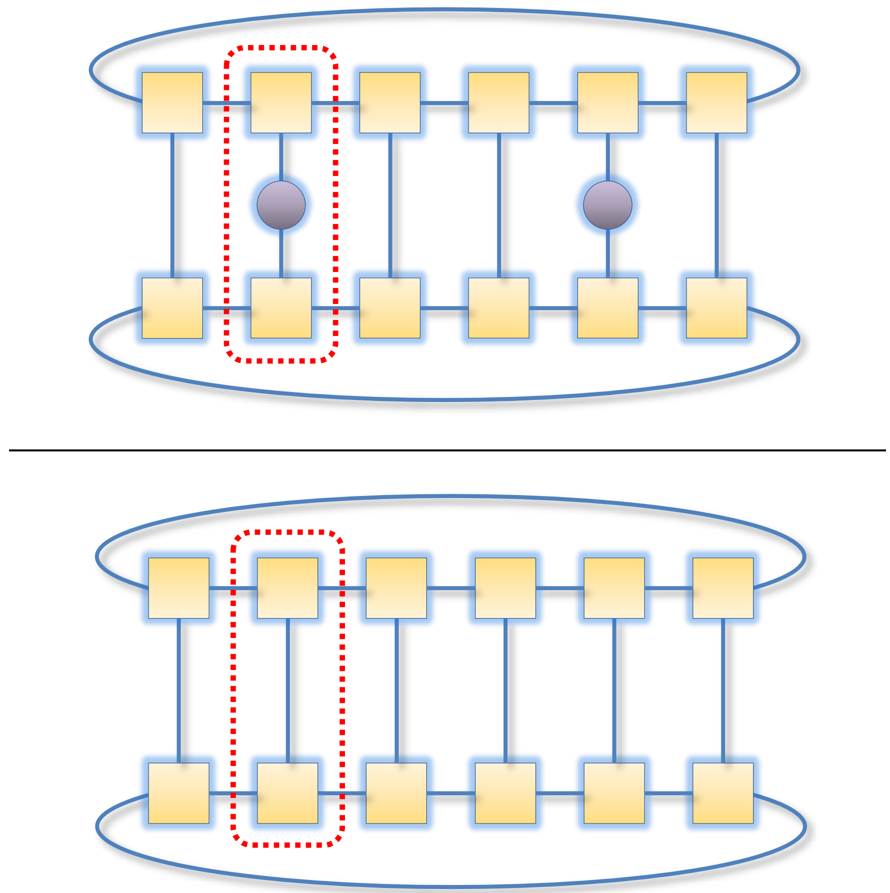

Use transfer matrices!

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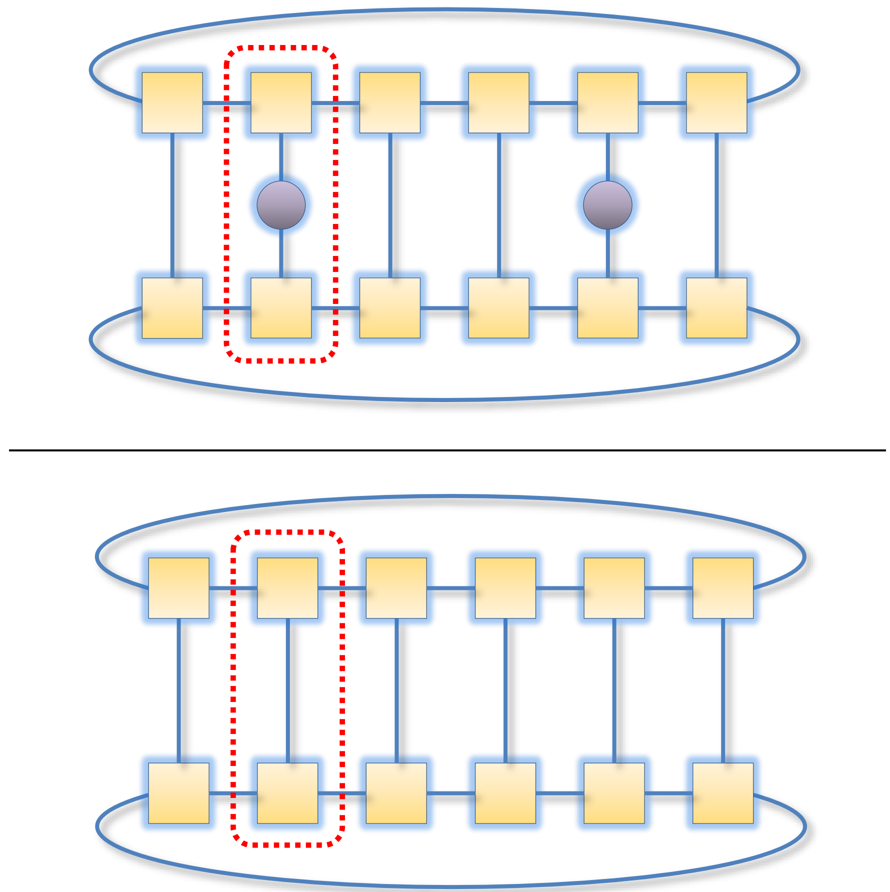
Correlation function:

$$\langle O_i O_j \rangle = \frac{\langle \psi | O_i O_j | \psi \rangle}{\langle \psi | \psi \rangle} =$$



$$\langle S_i^z S_j^z \rangle \sim e^{-|j-i|/\xi} \qquad \xi = \frac{1}{\ln(3)}$$

exponentially decay!

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Hidden string order:

… 0 0 1 0 -1 0 0 1 -1 0 0 0 1 0 -1 0 0 1 …

$$\langle S_i^z \prod_{k=i+1}^{j-1} e^{i\pi S_k^z} S_j^z \rangle = -\frac{4}{9}$$

Den Nijs & Rommelse, Phys. Rev. B 40, 4709 (1989);
Kennedy & Tasaki, Phys. Rev. B 45, 304 (1992).