TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

SFB 1143

Tensor Networks (SS2021)

Lecture 6: MPS injectivity

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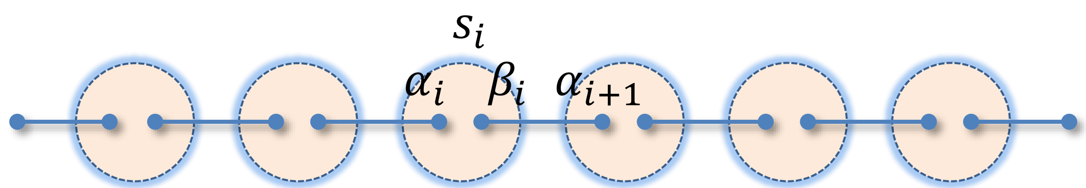
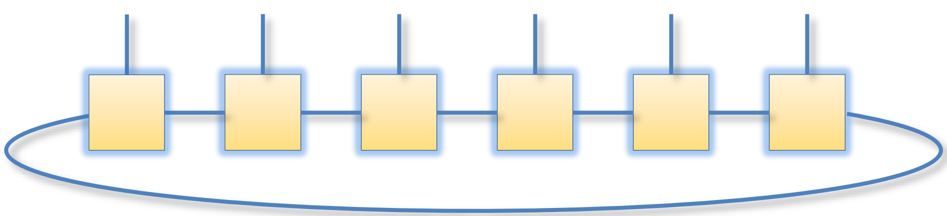
§ 1.6 AKLT model

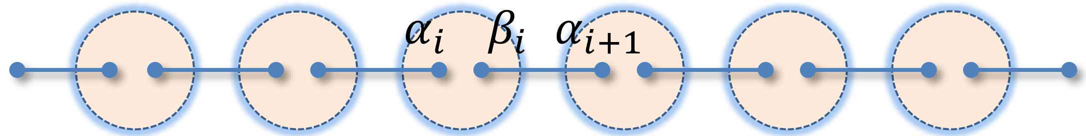


$$P_i = \sum_{s_i=1}^{d} \sum_{\alpha_i,\beta_i=1}^{D} B^{s_i}_{\alpha_i\beta_i} |s_i\rangle\langle\alpha_i,\beta_i| \qquad |I\rangle_{i,i+1} = \sum_{\beta_i,\alpha_{i+1}=1}^{D} R_{\beta_i,\alpha_{i+1}} |\beta_i,\alpha_{i+1}\rangle$$

$$|\psi_{\mathrm{PBC}}\rangle = \sum_{s_1,s_2,\ldots,s_N=1}^{d} \mathrm{Tr}\,(A^{s_1}A^{s_2}\cdots A^{s_N})|s_1,s_2,\ldots,s_N\rangle \qquad A^{s_i} = B^{s_i}R$$



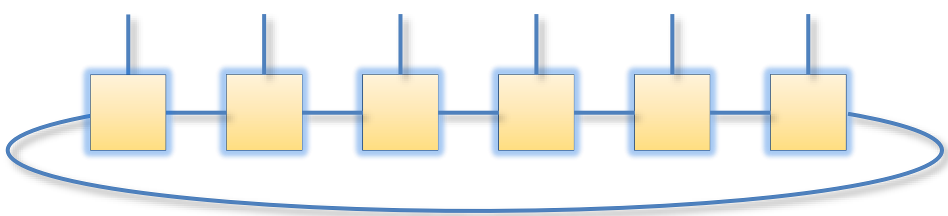
§ 1.7 MPS injectivity



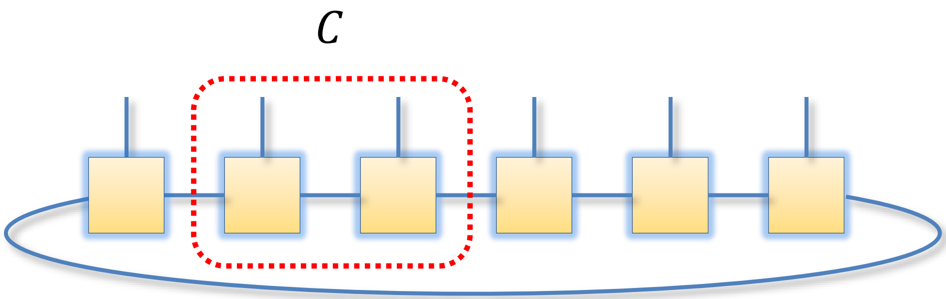
$$P_i = \sum_{s_i=1}^{d} \sum_{\alpha_i,\beta_i=1}^{D} A^{s_i}_{\alpha_i\beta_i} |s_i\rangle\langle\alpha_i,\beta_i| \qquad |I\rangle_{i,i+1} = \sum_{\beta_i,\alpha_{i+1}=1}^{D} \delta_{\beta_i,\alpha_{i+1}} |\beta_i,\alpha_{i+1}\rangle$$

maximally entangled state

$$|\psi_{\mathrm{PBC}}\rangle = \sum_{s_1,s_2,\ldots,s_N=1}^{d} \mathrm{Tr}\,(A^{s_1}A^{s_2}\cdots A^{s_N})|s_1,s_2,\ldots,s_N\rangle$$



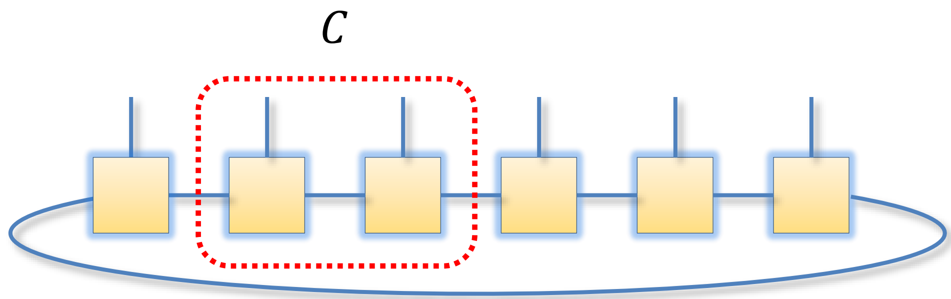
§ 1.7 MPS injectivity



$$|\psi\rangle = \sum_{\alpha,\beta=1}^{D} |\phi_{\alpha\beta}\rangle \otimes |\tilde{\phi}_{\alpha\beta}\rangle$$

Perez-Garcia, Verstraete, Wolf & Cirac, Quantum Inf. Comput. 7, 401 (2007).

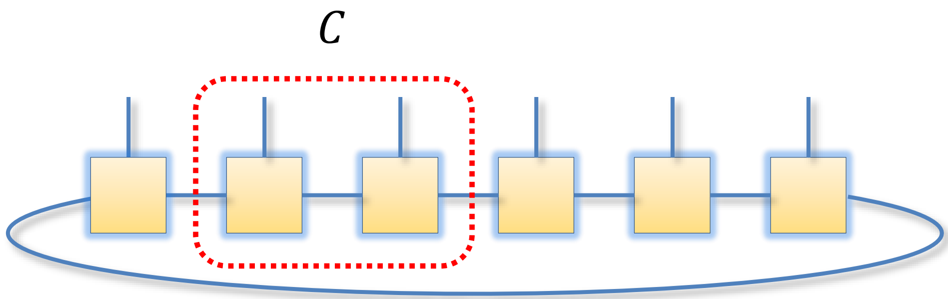
§ 1.7 MPS injectivity



$$|\psi\rangle = \sum_{\alpha,\beta=1}^{D} \underline{|\phi_{\alpha\beta}\rangle} \otimes |\tilde{\phi}_{\alpha\beta}\rangle$$

At most D^2 linearly independent vectors
(no matter how large the block C is)!

§ 1.7 MPS injectivity



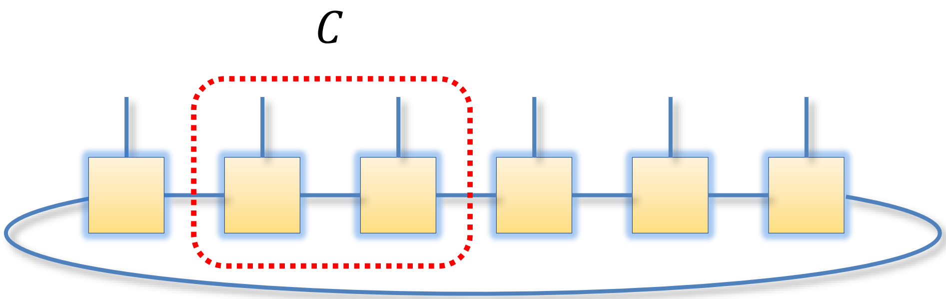
$$|\psi\rangle = \sum_{\alpha,\beta=1}^{D} |\phi_{\alpha\beta}\rangle \otimes |\tilde{\phi}_{\alpha\beta}\rangle$$

ρ_C: reduced density matrix of block C

At most D^2 linearly independent vectors (no matter how large the block C is)!

$\mathrm{Rank}(\rho_C) \leq D^2$ ⇒ ρ_C has a kernel if $d^{L_c} > D^2$.

§ 1.7 MPS injectivity



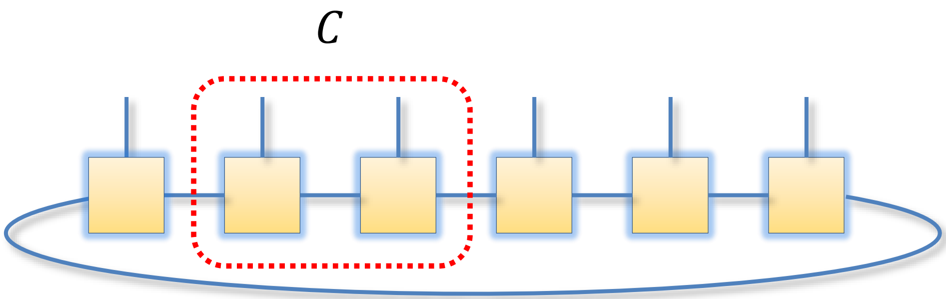
$$|\psi\rangle = \sum_{\alpha,\beta=1}^{D} |\phi_{\alpha\beta}\rangle \otimes |\tilde{\phi}_{\alpha\beta}\rangle$$

h_i: projector onto the kernel of ρ_C

$h_i \geq 0$

$h_i|\phi_{\alpha\beta}\rangle = 0$ ⇒ $h_i|\psi\rangle = 0$

§ 1.7 MPS injectivity



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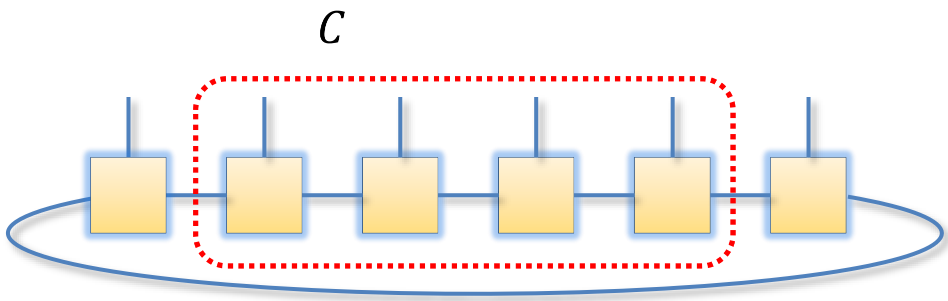
h_i: projector onto the kernel of ρ_C

$h_i|\phi_{\alpha\beta}\rangle = 0 \Rightarrow h_i|\psi\rangle = 0$

$h_i \geq 0$

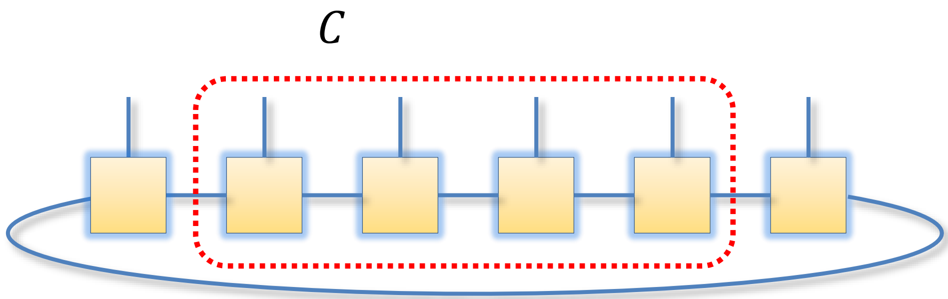
Parent Hamiltonian: $H = \sum_i h_i$

§ 1.7 MPS injectivity



Injective MPS: There exists a block size M such that $\mathrm{Rank}(\rho_C) = D^2$.

§ 1.7 MPS injectivity

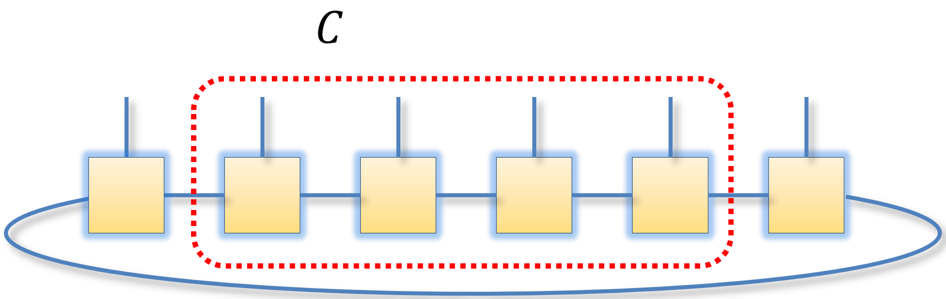


Injective MPS: There exists a block size M such that $\mathrm{Rank}(\rho_C) = D^2$.

smallest M: injectivity length

Example: spin-1 AKLT ($M = 2$), random MPS, ...

§ 1.7 MPS injectivity



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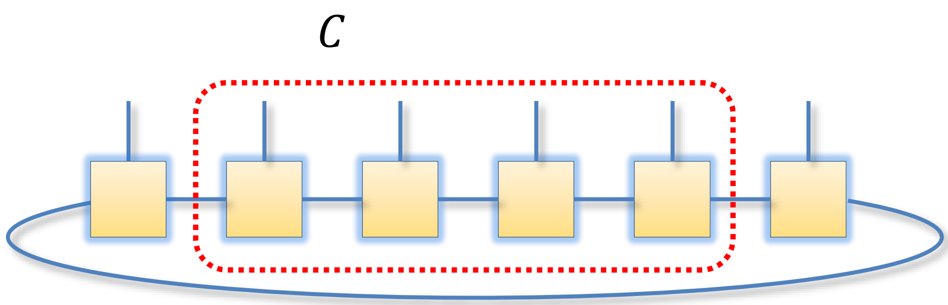
smallest M: injectivity length

Example: spin-1 AKLT ($M = 2$), random MPS, …

Quantum Wielandt Theorem: $M \leq (D^2 - d + 1)D^2$

Sanz, Perez-Garcia, Wolf & Cirac, IEEE Trans. Inf. Theory 56, 4668 (2010).

§ 1.7 MPS injectivity

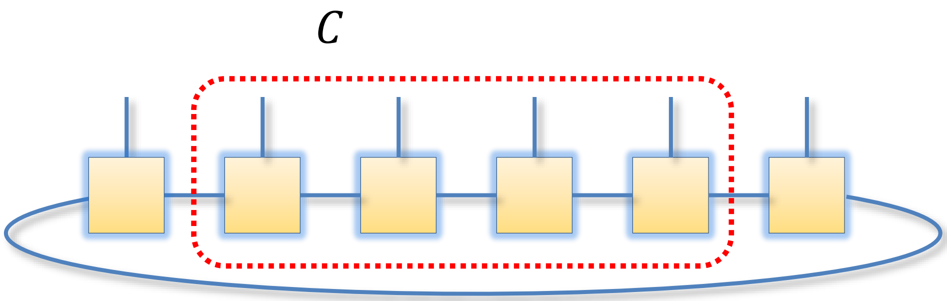


Injective MPS: There exists a block size M such that $\mathrm{Rank}(\rho_C) = D^2$.

- Consequence: The injective MPS is the unique gapped ground state of a $(M+1)$-local parent Hamiltonian.

h_i acts on $M+1$ successive sites

§ 1.7 MPS injectivity



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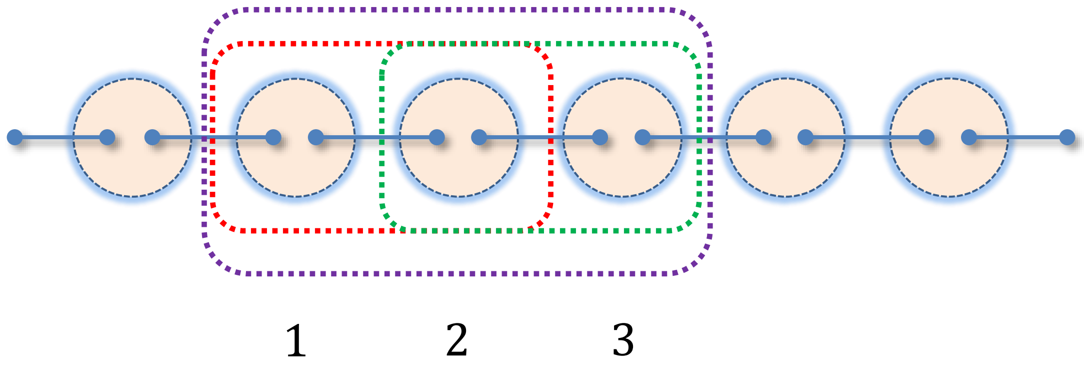
  → h_i acts on $M + 1$ successive sites

- Parent Hamiltonians with shorter-range interactions might exist (need to check intersection properties).

§ 1.7 MPS injectivity



Example: intersection property in spin-1 AKLT ($M = 2$)

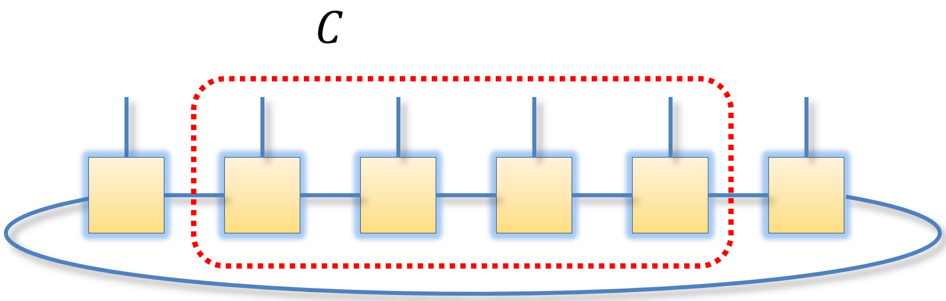
$V_1 = \{|\phi_\alpha^\perp\rangle_{12} \otimes |s_3\rangle\}$ → $\ker(\rho_{12})$

$V_2 = \{|s_1\rangle \otimes |\phi_\beta^\perp\rangle_{23}\}$ → $\ker(\rho_{23})$

$V_3 = \{|\phi_\gamma^\perp\rangle_{123}\}$ → $\ker(\rho_{123})$

$V_1 \cap V_2 = V_3$

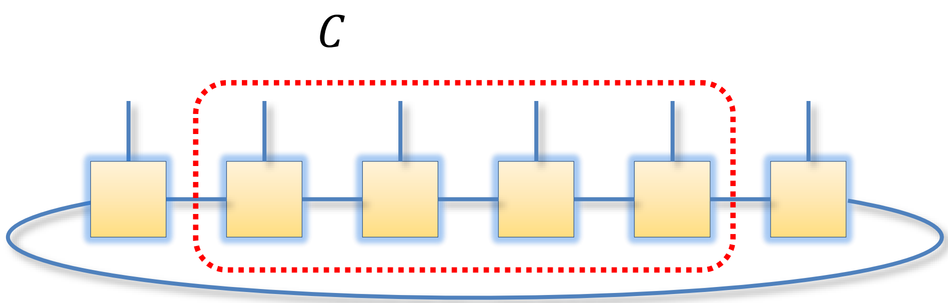
§ 1.7 MPS injectivity



Non-injective MPS: No matter how large M is, $\mathrm{Rank}(\rho_C) < D^2$.

Example: GHZ state

§ 1.7 MPS injectivity



Non-injective MPS: No matter how large M is, $\mathrm{Rank}(\rho_C) < D^2$.

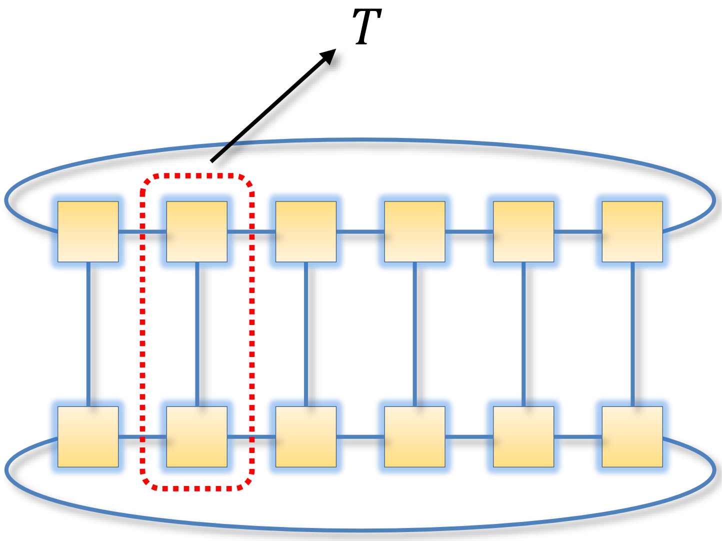
- Non-injective MPS can always be decomposed into a sum of injective MPS.

$$A^s = \begin{pmatrix} A_1^s & & \\ & A_2^s & \\ & & \ddots \end{pmatrix}$$

§ 1.7 MPS injectivity

Q: How do we check whether an MPS is injective in practice?

A: Check the spectrum of transfer matrix.



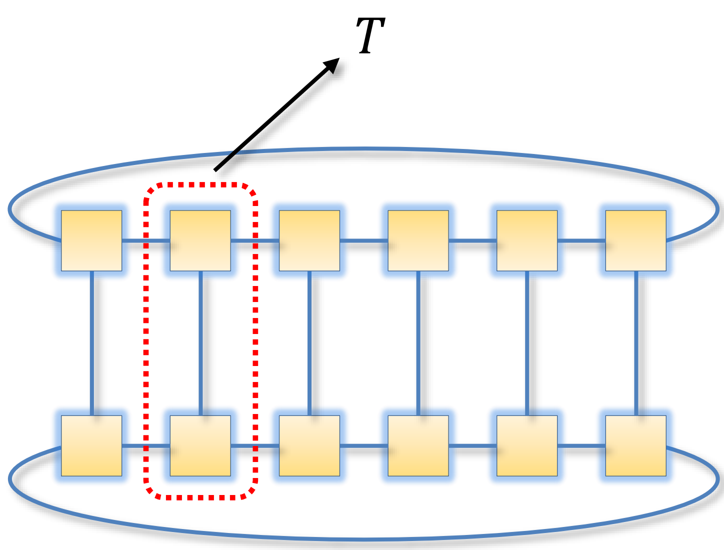
§ 1.7 MPS injectivity

Q: How do we check whether an MPS is injective in practice?

A: Check the spectrum of transfer matrix.

$$T|R\rangle = \lambda_{\max}|R\rangle$$

$$\langle L|T = \langle L|\lambda_{\max}$$



leading eigenvalue unique

$\Rightarrow$ MPS injective

§ 1.7 MPS injectivity

- Injective MPS always have exponentially decaying correlations.

$$\langle O_i O_j \rangle - \langle O_i \rangle \langle O_j \rangle \sim e^{-|j-i|/\xi}$$

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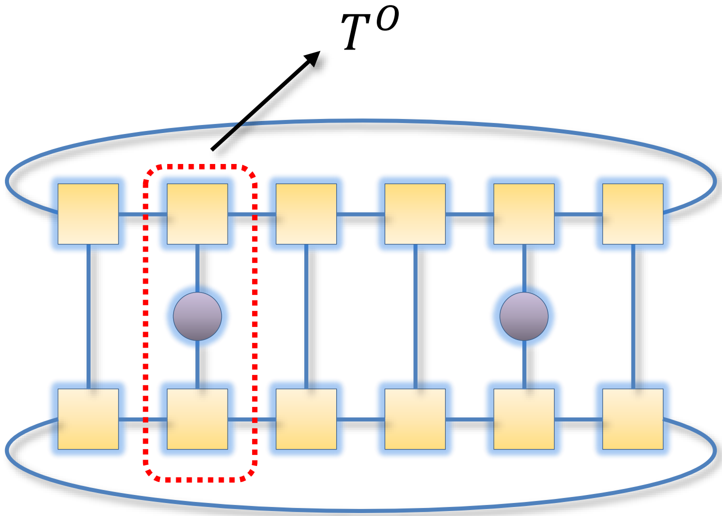
Use $T = \sum_{n=1}^{D^2} \lambda_n |R_n\rangle\langle L_n|$ and $\langle L_n | R_m \rangle = \delta_{nm}$:

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots$$



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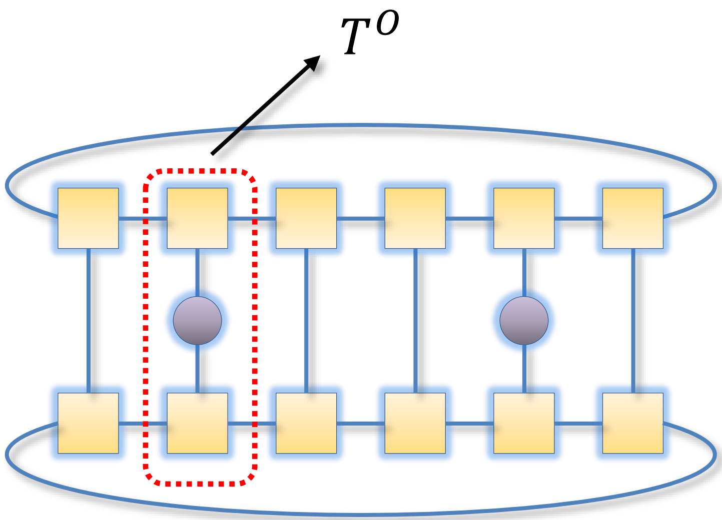
$$\langle O_i \rangle = \frac{\mathrm{tr}(T^O T^{N-1})}{\mathrm{tr}(T^N)}$$

$$\xrightarrow{N \to \infty} \frac{\langle L_1|T^O|R_1\rangle}{\lambda_1}$$



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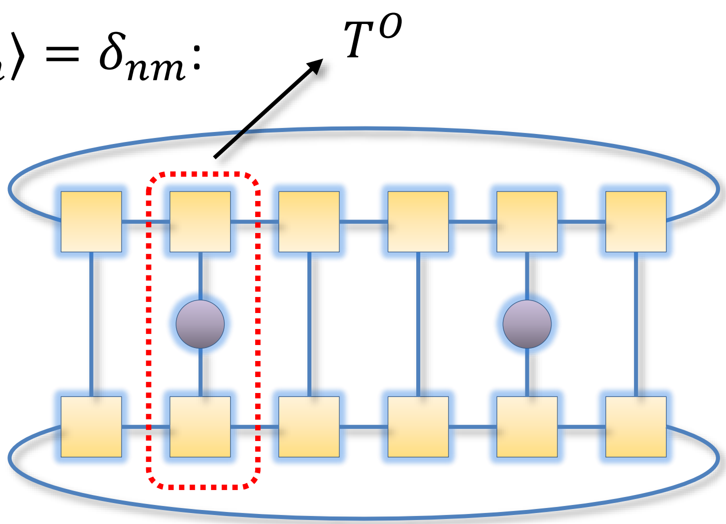


$$\langle O_i O_j \rangle = \frac{\mathrm{tr}(T^O T^{j-i-1} T^O T^{N-j+i-1})}{\mathrm{tr}(T^N)}$$

$$\xrightarrow{N \to \infty} \frac{\langle L_1 | T^O T^{j-i-1} T^O | R_1 \rangle}{\lambda_1^{j-i+1}}$$

$$\xrightarrow{|j-i| \to \infty} \left(\frac{\langle L_1 | T^O | R_1 \rangle}{\lambda_1} \right)^2 + \frac{\langle L_1 | T^O | R_2 \rangle \langle L_2 | T^O | R_1 \rangle}{\lambda_1 \lambda_2} \left(\frac{\lambda_2}{\lambda_1} \right)^{j-i}$$

(assume $|\lambda_2| > |\lambda_3|$)

§ 1.7 MPS injectivity

- Injective MPS always have exponentially decaying correlations.

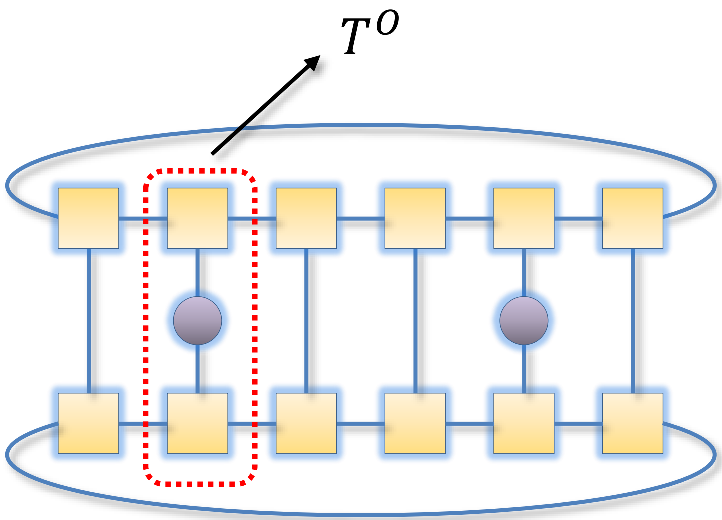
$$\langle O_i O_j \rangle - \langle O_i \rangle \langle O_j \rangle \sim e^{-|j-i|/\xi}$$

Use $T = \sum_{n=1}^{D^2} \lambda_n \, |R_n\rangle\langle L_n|$ and $\langle L_n | R_m \rangle = \delta_{nm}$:

$$\langle O_i O_j \rangle - \langle O_i \rangle \langle O_j \rangle \sim \text{const.} \times \left(\frac{\lambda_2}{\lambda_1}\right)^{j-i}$$

$$\sim e^{-|j-i|/\xi}$$

Correlation length: $\xi = 1/\ln\left|\frac{\lambda_1}{\lambda_2}\right|$



§ 1.7 MPS injectivity

Two-point correlation functions in MPS:

$$\langle O_i O_j \rangle \sim \begin{cases} \exp(-|j-i|/\xi) & \text{exponentially decay} \\ \text{const.} & \text{long-ranged} \end{cases}$$

- It's not possible to obtain powerlaw decaying correlations from MPS with finite D.

⇨ MPS have some deficiencies in describing critical systems.

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Review on MPS theory: Cirac, Perez-Garcia, Schuch & Verstraete, arXiv:2011.12127.