Posted: 17.10.2018
Tutorial: 17.10.2018
Due: 30.10.2018

Problem Set 1

1. **Heisenberg uncertainty principle** (3+2 points)

(a) The squared standard deviation of an operator $\hat{A}$ in a normalized quantum state $|\psi\rangle$ is defined by $\langle(\Delta\hat{A})^2\rangle$, where $\langle\cdots\rangle$ stands for the expectation value in $|\psi\rangle$ and $\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$.

For the position operator $\hat{x}$ and momentum operator $\hat{p}$, prove Heisenberg's uncertainty principle

$$\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p}^2)\rangle \geq \frac{\hbar^2}{4}\,.$$

Hint: Use the Cauchy-Schwarz inequality $\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle \geq |\langle\psi_1|\psi_2\rangle|^2$ ($|\psi_1\rangle$ and $|\psi_2\rangle$ are not necessarily normalized) and the commutation relation $[\hat{x},\hat{p}] = i\hbar$.

(b) The eigenstates of the harmonic oscillator Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$ satisfy $\hat{H}|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle$.

Calculate $\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p}^2)\rangle$ for each eigenstate $|n\rangle$ and verify that Heisenberg's uncertainty principle is not violated.

Hint: You may find it convenient to use the bosonic representation of the position and momentum operators.

2. **Path integral for a free particle** (5 points)

For a free particle in one dimension, the Hamiltonian is given by $\hat{H} = \frac{\hat{p}^2}{2m}$. Calculate the propagator

$$G(x',t';x,t) = \langle x'|e^{-\frac{i}{\hbar}\hat{H}(t'-t)}|x\rangle$$

by using the path integral approach.

Check your result: $G(x',t';x,t) = \sqrt{\frac{m}{2\pi i\hbar(t'-t)}}\exp\left[\frac{im(x'-x)^2}{2\hbar(t'-t)}\right]$.

3. **Partition function for a harmonic oscillator** (3+5 points)

For the harmonic oscillator in one dimension, the Hamiltonian is given by $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$. At finite temperature T, the partition function is define by $Z = \mathrm{Tr}(e^{-\beta\hat{H}})$, where $\beta = 1/T$.

(a) Derive the path integral form of the partition function by dividing the imaginary-time evolution into small time intervals.

(b) Calculate the partition function by performing Gaussian integrations.

Check your result: $Z = 1/(e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2})$.