Problem Set 2

1. **Majorana edge mode** (2+2+2 points)

Consider a 1D chain with L sites and open boundary condition with the Hamiltonian

$$\hat{H} = i\Delta \sum_{j=1}^{L-1} \hat{c}_{2j}\hat{c}_{2j+1} + i\mu \sum_{j=1}^{L} \hat{c}_{2j-1}\hat{c}_{2j},$$

where $\hat{c}_j$ are Majorana fermions satisfying $\{\hat{c}_j, \hat{c}_l\} = 2\delta_{jl}$ and Δ, μ are real parameters. This model is a special case of the Kitaev's Majorana chain (see cond-mat/0010440), which describes a topological (trivial) superconductor with (without) Majorana edge modes for $|\mu/\Delta| < 1$ ($|\mu/\Delta| > 1$).

(a) For $\mu = 2$ and $\Delta = 1$, diagonalize $\hat{H}$ numerically for $L = 6, 10, 16, 20, 50$. Plot the single-particle excitation energies of $\hat{H}$. Verify that the Hamiltonian has a gap and does not support Majorana edge modes.

(b) For $\mu = 1$ and $\Delta = 2$, plot the single-particle excitation energies of $\hat{H}$ for $L = 6, 10, 16, 20, 50$. Verify that the Hamiltonian has a gap and two Majorana edge modes.

Hint: For (a) and (b), you may write the Hamiltonian as $\hat{H} = \frac{i}{4}\hat{c}^T A \hat{c}$, where A is a $2L \times 2L$ real antisymmetric matrix whose eigenvalues $\pm i\varepsilon_m$ come in pairs. You may then show that the Hamiltonian is diagonalized as $\hat{H} = \sum_{m=1}^{L} \varepsilon_m d_m^\dagger d_m$, where ε_m are single-particle excitation energies and d_m are fermionic operators satisfying $\{d_m, d_{m'}^\dagger\} = \delta_{mm'}$.

(c) In the topological phase with $|\mu/\Delta| < 1$, show that the exponentially localized zero-energy Majorana edge modes $\hat{b}_\mathrm{L}$ and $\hat{b}_\mathrm{R}$ take the form $\hat{b}_\mathrm{L} \propto \sum_{j=1}^{L} (\mu/\Delta)^{j-1}\hat{c}_{2j-1}$ and $\hat{b}_\mathrm{R} \propto \sum_{j=1}^{L} (\mu/\Delta)^{L-j}\hat{c}_{2j}$ in the thermodynamic limit $L \to \infty$.

Hint: You may prove this by showing that b_L and b_R commute with the Hamiltonian for $L \to \infty$.

2. **SU(2) spin coherent state** (2+2+3 points)

For a single spin-S, the SU(2) spin coherent state is defined by $|\vec{\Omega}\rangle = \frac{1}{\sqrt{(2S)!}}(z_1\hat{a}^\dagger + z_2\hat{b}^\dagger)^{2S}|0\rangle$, where $\hat{a}^\dagger$, $\hat{b}^\dagger$ are Schwinger bosons, $(z_1, z_2) = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}e^{i\phi})$, and the O(3) unit vector $\vec{\Omega} = (\Omega^x, \Omega^y, \Omega^z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The spin operators are represented by Schwinger boson operators, $\hat{S}^x = \frac{1}{2}(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})$, $\hat{S}^y = \frac{1}{2i}(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})$, and $\hat{S}^z = \frac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})$, with the constraint $\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} = 2S$. The physical spin states $|S,m\rangle$ are related to the Schwinger boson occupation basis via $|S,m\rangle = \frac{(\hat{a}^\dagger)^{S+m}(\hat{b}^\dagger)^{S-m}}{\sqrt{(S+m)!(S-m)!}}|0\rangle$ ($m = -S, -S+1, \ldots, S$).

(a) Prove that the spin coherent state satisfies the overcompleteness relation

$$\frac{2S+1}{4\pi}\int d\vec{\Omega}\ |\vec{\Omega}\rangle\langle\vec{\Omega}| = \sum_{m=-S}^{S} |S,m\rangle\langle S,m| = 1,$$

where $\int d\vec{\Omega} = \int_0^\pi \sin\theta\ d\theta \int_0^{2\pi} d\phi$.

(b) Show that the expectation values of the spin operators in the spin coherent state are given by

$$\langle\vec{\Omega}|\hat{S}^a|\vec{\Omega}\rangle = S\Omega^a, \qquad (a = x, y, z).$$

(c) The quantum spin-S Heisenberg model on a lattice is defined by $\hat{H} = J\sum_{\langle i,j\rangle} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j$. By discretizing the imaginary-time evolution into small time-slices and inserting the overcompleteness relation of spin coherent states, show that the partition function $Z = \mathrm{Tr} e^{-\beta \hat{H}}$ has the following path integral formulation:

$$Z = \int_{\vec{\Omega}(0)=\vec{\Omega}(\beta)} \mathcal{D}\vec{\Omega}(\tau)\ e^{-S[\vec{\Omega}]},$$

where

$$S[\vec{\Omega}] = \int_0^\beta d\tau \ \left[iS \sum_j (1-\cos\theta_j)\dot{\phi}_j + JS^2 \sum_{\langle j,l \rangle} \vec{\Omega}_j \cdot \vec{\Omega}_l \right].$$

3. **1D classical XY model** (2+3 points)

Consider the partition function of the 1D classical XY model

$$Z = \int_0^{2\pi} \prod_{i=1}^{L} \frac{d\theta_i}{2\pi} e^{-\beta H},$$

where $\beta = 1/T$ and $H = -J\sum_{i=1}^{L} \mathbf{I}_i \cdot \mathbf{I}_{i+1}$ with $\mathbf{I}_i = I(\cos\theta_i, \sin\theta_i)$ and periodic boundary condition ($\theta_1 = \theta_{L+1}$). The partition function can be represented as

$$Z = \int_0^{2\pi} \prod_{i=1}^{L} \frac{d\theta_i}{2\pi} \mathrm{T}(\theta_1,\theta_2)\mathrm{T}(\theta_2,\theta_3)\cdots \mathrm{T}(\theta_L,\theta_1),$$

where $\mathrm{T}(\theta_i,\theta_{i+1}) = e^{\beta J I^2 \cos(\theta_i - \theta_{i+1})}$ is the so-called transfer matrix.

(a) Calculate the partition function Z.

Hint: It might be convenient to diagonalize the transfer matrix as $\mathrm{T}(\theta,\theta') = \sum_{m=-\infty}^{+\infty} A_m e^{im(\theta-\theta')}$. To achieve this you may use the identity $e^{z\cos\theta} = I_0(z) + 2\sum_{m=1}^{\infty} I_m(z)\cos(m\theta)$, where $I_m(z) = \sum_{n=0}^{\infty} \frac{1}{(n+m)!n!}\left(\frac{z}{2}\right)^{2n+m}$ is the modified Bessel function of the first kind.

(b) Calculate the correlation function

$$\langle \mathbf{I}_j \cdot \mathbf{I}_l \rangle = \frac{1}{Z}\int_0^{2\pi} \prod_{i=1}^{L} \frac{d\theta_i}{2\pi} (\mathbf{I}_j \cdot \mathbf{I}_l) e^{-\beta H}$$

by using the transfer matrix approach. Show that the correlation function $\langle \mathbf{I}_j \cdot \mathbf{I}_l \rangle$ decays exponentially at large distance $|j-l| \gg 1$,

$$\lim_{L\to\infty} \langle \mathbf{I}_j \cdot \mathbf{I}_l \rangle \sim e^{-|j-l|/\xi}$$

for any finite temperature $T > 0$. Determine the correlation length ξ and verify that it agrees with the high-temperature expansion result when T is large.