Problem Set 3

1. **Goldstone boson** (3+2+3 points)

Consider the O(2) quantum rotor model on a three-dimensional cubic lattice (c.f. Lecture note PT-2.pdf). The Goldstone boson excitation above the symmetry breaking ground state is described by the partition function

$$Z = \int \mathcal{D}\theta \ \exp\left[-\sum_{\mathbf{k},i\omega_n} \frac{C}{2}(\omega_n^2 + \omega_{\mathbf{k}}^2)\theta(-\mathbf{k},-i\omega_n)\theta(\mathbf{k},i\omega_n)\right]$$

with $\omega_n = \frac{2\pi n}{\beta}$ ($n \in \mathbb{Z}$ and $\beta = 1/T$) is the bosonic Matsubara frequency and $\omega_{\mathbf{k}} = \sqrt{\frac{2J}{C}\sum_{l=1}^{3}(1-\cos k_l a)}$ is the dispersion relation of the Goldstone boson, where k_l are lattice momenta within the first Brillouin zone (FBZ), $k_l \in (-\frac{\pi}{a}, \frac{\pi}{a}]$. The fluctuation field $\delta\theta(\mathbf{r},\tau)$ is real so that its Fourier components satisfy $\theta(-\mathbf{k},-i\omega_n) = \theta^*(\mathbf{k},i\omega_n)$.

(a) Show that the contribution of the Goldstone boson to the free energy is given by

$$F = -\frac{1}{\beta}\ln Z = \frac{1}{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}} + \frac{1}{\beta}\sum_{\mathbf{k}}\ln(1-e^{-\beta\omega_{\mathbf{k}}}).$$

Hint: When performing the ω_n-summation, you may use the identity $\frac{\sinh(z)}{z} = \prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2 n^2}\right)$. After evaluating the Gaussian integral, you may encounter divergent terms in the free energy. However, these terms should be canceled by another constant coming from the path integral measure. This is based on the physical consideration that the free energy has no divergences and should recover the results from the Hamiltonian formulation with $\hat{H} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}(\hat{b}_{\mathbf{k}}^\dagger\hat{b}_{\mathbf{k}} + \frac{1}{2})$.

(b) Show that the contribution of the Goldstone boson to the specific heat is given by

$$C_V = -T\frac{\partial^2 F}{\partial T^2} = \frac{1}{T^2}\sum_{\mathbf{k}}\frac{e^{\omega_{\mathbf{k}}/T}}{(e^{\omega_{\mathbf{k}}/T}-1)^2}\omega_{\mathbf{k}}^2.$$

(c) How does the specific heat scale with T at low temperature ($T \ll \max_{\mathbf{k}\in\mathrm{FBZ}}\omega_{\mathbf{k}}$)?

Hint: The calculation is similar to the derivation of the Debye's T^3 law for acoustic phonons arising from lattice vibrations.

2. **Correlation functions for $d = 2$ massless bosonic field** (3+2 points)

Consider the partition function of $d = 2$ massless bosonic field

$$Z = \int \mathcal{D}\phi(\mathbf{r}) \ e^{-\int d^2\mathbf{r}\frac{1}{2}[\nabla\phi(\mathbf{r})]^2},$$

which is an important theory in several research fields, such as statistical physics and string theory. During the lecture, this theory appears as the effective theory for describing the $d = 2$ classical XY model *below* the KT transition temperature.

(a) Show that the correlation function $G(\mathbf{r}_1,\mathbf{r}_2) = \langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle$ is given by

$$G(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{Z}\int \mathcal{D}\phi(\mathbf{r}) \ \phi(\mathbf{r}_1)\phi(\mathbf{r}_2)e^{-\int d^2\mathbf{r}\frac{1}{2}[\nabla\phi(\mathbf{r})]^2} = -\frac{1}{2\pi}\ln\frac{|\mathbf{r}_1-\mathbf{r}_2|}{a},$$

where a is a cutoff.

Hint: You may show that $\langle \phi_{\mathbf{k}_1}\phi_{-\mathbf{k}_2}\rangle = \delta_{\mathbf{k}_1,\mathbf{k}_2}\frac{1}{|\mathbf{k}_1|^2}$, where $\phi_{\mathbf{k}}$ is the Fourier component of $\phi(\mathbf{r})$, i.e., $\phi(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\phi_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}$. Then, $G(\mathbf{r}_1,\mathbf{r}_2) = \langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle$ can be obtained by a Fourier transformation of $\langle \phi_{\mathbf{k}_1}\phi_{-\mathbf{k}_2}\rangle$. In this way, you would see that $G(\mathbf{r}) = \langle\phi(\mathbf{r})\phi(0)\rangle$ satisfies the Laplace equation $\nabla^2_{\mathbf{r}}G(\mathbf{r}) = -\delta(\mathbf{r})$, where $G(\mathbf{r})$ can be obtained after integrating over $\mathbf{r}$ within a disk.

(b) Show that the correlation function $\langle e^{i\alpha\phi(\mathbf{r}_1)}e^{-i\alpha\phi(\mathbf{r}_2)}\rangle$ is given by

$$\langle e^{i\alpha\phi(\mathbf{r}_1)}e^{-i\alpha\phi(\mathbf{r}_2)}\rangle = \frac{1}{Z}\int\mathcal{D}\phi(\mathbf{r})\; e^{i\alpha\phi(\mathbf{r}_1)}e^{-i\alpha\phi(\mathbf{r}_2)}e^{-\int d^2\mathbf{r}\frac{1}{2}[\nabla\phi(\mathbf{r})]^2} = \left(\frac{a}{|\mathbf{r}_1-\mathbf{r}_2|}\right)^{\alpha^2/2\pi}.$$

This proves that the scaling dimension of the field $e^{i\alpha\phi(\mathbf{r})}$ is $\alpha^2/4\pi$, which is essential for understanding the RG relevance/irrelevance of the cosine potential in the Sine-Gordon model.

Hint: You may use the result in (a). This can be done by using a very useful identity $\langle e^{i\alpha[\phi(\mathbf{r}_1)-\phi(\mathbf{r}_2)]}\rangle = e^{-\frac{1}{2}\alpha^2\langle[\phi(\mathbf{r}_1)-\phi(\mathbf{r}_2)]^2\rangle}$ for free boson theories. This identity can be proved by using the Wick's theorem and may be viewed as a generalization of $\int_{-\infty}^{+\infty}e^{-ax^2+ibx}dx = \sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}} = \sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{2}\langle x^2\rangle}$.

3. **Bogoliubov transformation** (2+3 points)

During the lecture, the Bogoliubov's theory of superfluidity has been formulated in terms of path integral. There is an equivalent formulation in terms of Hamiltonian defined with operators. The Hamiltonian describing the Bogoliubov quasiparticles is given by (c.f. page 6 in boson-2.pdf)

$$\hat{H} = \sum_{\mathbf{k}(\neq 0)}\left[\left(\frac{\mathbf{k}^2}{2m} + g\psi_0^2\right)\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{2}g\psi_0^2(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} + \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}})\right],$$

where $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ are bosonic annihilation and creation operators, respectively.

To diagonalize the Hamiltonian, one needs to use the Bogoliubov transformation, which introduces a new set of bosonic operators

$$\hat{\alpha}_{\mathbf{k}} = u_{\mathbf{k}}\hat{a}_{\mathbf{k}} + v_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}, \qquad \hat{\alpha}^\dagger_{\mathbf{k}} = u^*_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} + v^*_{\mathbf{k}}\hat{a}_{-\mathbf{k}},$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are numbers. For the present Hamiltonian, $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ can be chosen as real numbers satisfying $u_{\mathbf{k}} = u_{-\mathbf{k}}$ and $v_{\mathbf{k}} = v_{-\mathbf{k}}$.

(a) Show that $\hat{\alpha}_{\mathbf{k}}$ and $\hat{\alpha}^\dagger_{\mathbf{k}}$ satisfy the standard bosonic commutation relations, $[\hat{\alpha}_{\mathbf{k}},\hat{\alpha}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$ if $|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2 = 1$.

(b) Find the proper choice of $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ which diagonalizes the Hamiltonian,

$$\hat{H} = \sum_{\mathbf{k}(\neq 0)}\omega_{\mathbf{k}}\hat{\alpha}^\dagger_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} + \text{const.}$$

Check that the dispersion relation $\omega_{\mathbf{k}}$ agrees with the path integral result $\omega_{\mathbf{k}} = \sqrt{\frac{\mathbf{k}^2}{2m}\left(\frac{\mathbf{k}^2}{2m} + 2g\psi_0^2\right)}$.