WS2018/19: Many-Body Theory in Condensed Matter
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Problem Set 4

1. **AKLT model** (2+2+3 points)

The ground state of the spin-1 AKLT-model in a periodic chain with N sites has the following matrix-product state (MPS) form:

$$|\Psi\rangle = \sum_{s_1,\ldots,s_N} \mathrm{Tr}(A^{s_1}\ldots A^{s_N})|s_1,\ldots,s_N\rangle\,,$$

where $s_j = +1, 0, -1$ and

$$A^{+1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad A^{0} = \frac{1}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad A^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}.$$

(a) Compute the transfer matrix

$$T_{(\alpha_l,\alpha'_l),(\alpha_{l+1},\alpha'_{l+1})} = \sum_{s_l}(A^{s_l}_{\alpha'_l,\alpha'_{l+1}})^* A^{s_l}_{\alpha_l,\alpha_{l+1}}\,,$$

and verify that its eigenvalues are $3/4$, $-1/4$, $-1/4$, $-1/4$.

(b) Compute the transfer matrices for $\hat{S}_z$ and $e^{i\pi\hat{S}^z}$, using

$$T^{\hat{O}}_{(\alpha_l,\alpha'_l),(\alpha_{l+1},\alpha'_{l+1})} = \sum_{s_l,s'_l}(A^{s_l}_{\alpha_l,\alpha_{l+1}})^* A^{s'_l}_{\alpha'_l,\alpha'_{l+1}}\langle s_l|\hat{O}|s'_l\rangle\,.$$

(c) Calculate spin-spin and string correlation functions

$$C^{zz}_{ij} \equiv \frac{\langle\Psi|\hat{S}^z_i\hat{S}^z_j|\Psi\rangle}{\langle\Psi|\Psi\rangle},$$
$$C^{\mathrm{string}}_{ij} \equiv -\frac{\langle\Psi|\hat{S}^z_i\prod_{l=i+1}^{j-1}e^{i\pi\hat{S}^z_l}\hat{S}^z_j|\Psi\rangle}{\langle\Psi|\Psi\rangle}.$$

Note that the string correlation function characterizes the hidden "dilute" antiferromagnetic order. How do these correlation functions behave asymptotically ($\lim_{|j-i|\to\infty}\lim_{N\to\infty}$)?

Check your result: $C^{zz}_{ij}\sim e^{-|i-j|/\xi}$, with $\xi = 1/\ln 3$ and $C^{\mathrm{string}}_{ij} = \frac{4}{9}$.

2. **Correlation function in the spin-1/2 XY chain** (3+1 points)

Consider a spin-1/2 XY chain with periodic boundary condition

$$\hat{H}_{\mathrm{PBC}} = -J\sum_{j=1}^{N}(\hat{S}^x_j\hat{S}^x_{j+1} + \hat{S}^y_j\hat{S}^y_{j+1}),$$

where N is even and $J>0$.

(a) Follow the exact solution of $\hat{H}_{\mathrm{PBC}}$ via Jordan-Wigner transformation. Calculate the longitudinal spin-spin correlation function

$$C^{zz}_{ij} \equiv \langle\Psi|\hat{S}^z_i\hat{S}^z_j|\Psi\rangle,$$

where $|\Psi\rangle$ is the (normalized) ground state of $\hat{H}_{\mathrm{PBC}}$, written in terms of Jordan-Wigner fermions as $|\Psi\rangle = \prod_{|k|<\frac{\pi}{2}}\hat{d}^\dagger_k|0\rangle$. When $N/2$ is even (odd), the ground state $|\Psi\rangle$ appears in the Neveu-Schwarz

(Ramond) sector with allowed momenta $k = \pm\frac{\pi}{N}, \pm\frac{3\pi}{N}, \ldots, \pm\frac{(N-1)\pi}{N}$ ($k = 0, \pm\frac{2\pi}{N}, \pm\frac{4\pi}{N}, \ldots, \pm\frac{(N-2)\pi}{N}, \pi$). How does C_{ij}^{zz} decay asymptotically ($\lim_{|j-i|\to\infty} \lim_{N\to\infty}$)?

Hint: Represent spin operators in C_{ij}^{zz} by using Jordan-Wigner fermions and use Wick's theorem.

(b) The transverse spin-spin correlation function $C_{ij}^{xx} \equiv \langle\Psi|\hat{S}_i^x \hat{S}_j^x|\Psi\rangle$ is not easy to calculate by using the above approach. Why?

3. **Spin-1/2 XY chain with open boundaries** (3+2+2 points)

Consider a spin-1/2 XY chain with open boundary condition, defined by the Hamiltonian

$$\hat{H}_{\text{OBC}} = -J \sum_{j=1}^{N-1} (\hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y),$$

where N is even and $J > 0$.

(a) Map $\hat{H}_{\text{OBC}}$ to a quadratic Hamiltonian of fermions by using the Jordan-Wigner transformation. Diagonalize the fermionic Hamiltonian.

Hint: You may diagonalize the fermionic Hamiltonian by using $\hat{c}_j = \sqrt{\frac{2}{N+1}} \sum_{m=1}^{N} \sin(\frac{\pi m}{N+1} j) \hat{d}_m$. Note that the inverse transformation is given by $\hat{d}_m = \sqrt{\frac{2}{N+1}} \sum_{j=1}^{N} \sin(\frac{\pi m}{N+1} j) \hat{c}_j$, where $\hat{d}_m$ satisfies $\{\hat{d}_m, \hat{d}_{m'}^\dagger\} = \delta_{mm'}$.

(b) Determine the ground-state energy E_0 and the eigenenergy E_1 of the first excited state. How does the energy difference $E_1 - E_0$ scale with the system size N?

(c) How does the ground-state wave function look like in the original spin basis?

Check your result: $|\Psi\rangle = \sum_{1\le x_1 < x_2 < \cdots < x_{N/2} \le N} \Psi(x_1, x_2, \ldots, x_{N/2}) \hat{\sigma}_{x_1}^+ \hat{\sigma}_{x_2}^+ \cdots \hat{\sigma}_{x_{N/2}}^+ |\downarrow, \downarrow \ldots, \downarrow\rangle$, where $\Psi(x_1, x_2, \ldots, x_{N/2}) \propto \prod_{i=1}^{N/2} \sin\frac{\pi x_i}{N+1} \prod_{1\le j<l\le N/2} \left(\cos\frac{\pi x_j}{N+1} - \cos\frac{\pi x_l}{N+1}\right)$.