Problem Set 3: Solution

1. **Goldstone boson** (3+2+3 points)

Consider the O(2) quantum rotor model on a three-dimensional cubic lattice (c.f. Lecture note PT-2.pdf). The Goldstone boson excitation above the symmetry breaking ground state is described by the partition function

$$Z = \int \mathcal{D}\theta \ \exp\left[-\sum_{\mathbf{k}, i\omega_n} \frac{C}{2}(\omega_n^2 + \omega_{\mathbf{k}}^2)\theta(-\mathbf{k}, -i\omega_n)\theta(\mathbf{k}, i\omega_n) \right]$$

with $\omega_n = \frac{2\pi n}{\beta}$ ($n \in \mathbb{Z}$ and $\beta = 1/T$) is the bosonic Matsubara frequency and $\omega_{\mathbf{k}} = \sqrt{\frac{2J}{C}\sum_{l=1}^{3}(1-\cos k_l a)}$ is the dispersion relation of the Goldstone boson, where k_l are lattice momenta within the first Brillouin zone (FBZ), $k_l \in (-\frac{\pi}{a}, \frac{\pi}{a}]$. The fluctuation field $\delta\theta(\mathbf{r}, \tau)$ is real so that its Fourier components satisfy $\theta(-\mathbf{k}, -i\omega_n) = \theta^*(\mathbf{k}, i\omega_n)$.

(a) Show that the contribution of the Goldstone boson to the free energy is given by

$$F = -\frac{1}{\beta}\ln Z = \frac{1}{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}} + \frac{1}{\beta}\sum_{\mathbf{k}}\ln(1 - e^{-\beta\omega_{\mathbf{k}}}).$$

(b) Show that the contribution of the Goldstone boson to the specific heat is given by

$$C_V = -T\frac{\partial^2 F}{\partial T^2} = \frac{1}{T^2}\sum_{\mathbf{k}}\frac{e^{\omega_{\mathbf{k}}/T}}{(e^{\omega_{\mathbf{k}}/T}-1)^2}\omega_{\mathbf{k}}^2.$$

(c) How does the specific heat scale with T at low temperature ($T \ll \max_{\mathbf{k}\in \text{FBZ}}\omega_{\mathbf{k}}$)?

Solution: (a) Since $\theta(-\mathbf{k}, -i\omega_n) = \theta^*(\mathbf{k}, i\omega_n)$, we divide them into the real and imaginary parts

$$\begin{aligned} \theta(\mathbf{k}, i\omega_n) &= \text{Re}\theta(\mathbf{k}, i\omega_n) + i\text{Im}\theta(\mathbf{k}, i\omega_n), \\ \theta(-\mathbf{k}, -i\omega_n) &= \text{Re}\theta(\mathbf{k}, i\omega_n) - i\text{Im}\theta(\mathbf{k}, i\omega_n), \end{aligned}$$

then

$$\begin{aligned} Z &= \int \mathcal{D}\theta \ \exp\left[-\sum_{\mathbf{k}, i\omega_n} \frac{C}{2}(\omega_n^2 + \omega_{\mathbf{k}}^2)\theta(-\mathbf{k}, -i\omega_n)\theta(\mathbf{k}, i\omega_n) \right] \\ &= \mathcal{N}\int \prod_{\mathbf{k}, i\omega_n (\text{half } \mathbf{k})} d\text{Re}\theta(\mathbf{k}, i\omega_n) d\text{Im}\theta(\mathbf{k}, i\omega_n) \\ &\quad \times \exp\left\{ -\sum_{\mathbf{k}, i\omega_n (\text{half } \mathbf{k})} C(\omega_n^2 + \omega_{\mathbf{k}}^2)\left[(\text{Re}\theta(\mathbf{k}, i\omega_n))^2 + (\text{Im}\theta(\mathbf{k}, i\omega_n))^2 \right] \right\} \quad \Leftarrow \text{use } \int_{-\infty}^{\infty} dx \ e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \\ &= \mathcal{N}\prod_{\mathbf{k}, i\omega_n (\text{half } \mathbf{k})} \frac{\pi}{C(\omega_n^2 + \omega_{\mathbf{k}}^2)} \\ &= \mathcal{N}\prod_{\mathbf{k}, i\omega_n} \sqrt{\frac{\pi}{C(\omega_n^2 + \omega_{\mathbf{k}}^2)}}, \end{aligned}$$

where $\mathcal{N}$ is an overall constant after changing the integration measure. Then, the free energy is given by

$$
\begin{aligned}
F &= -\frac{1}{\beta}\ln Z \\
&= -\frac{1}{\beta}\ln\mathcal{N} - \frac{1}{2\beta}\sum_{\mathbf{k},i\omega_n}\ln\frac{\pi}{C(\omega_n^2+\omega_{\mathbf{k}}^2)} \\
&= -\frac{1}{\beta}\ln\mathcal{N}' + \frac{1}{2\beta}\sum_{\mathbf{k},i\omega_n}\ln(\omega_n^2+\omega_{\mathbf{k}}^2), \qquad (1)
\end{aligned}
$$

where the summation over bosonic Matsubara frequency can be carried out

$$
\begin{aligned}
&\sum_{i\omega_n}\ln(\omega_n^2+\omega_{\mathbf{k}}^2) \\
=\;& \ln\omega_{\mathbf{k}}^2 + 2\sum_{n=1}^{\infty}\ln\left(\frac{4\pi^2n^2}{\beta^2}+\omega_{\mathbf{k}}^2\right) \\
=\;& 2\ln\omega_{\mathbf{k}} + 2\sum_{n=1}^{\infty}\ln\left[\frac{4\pi^2n^2}{\beta^2}\left(1+\frac{\beta^2\omega_{\mathbf{k}}^2}{4\pi^2n^2}\right)\right] \\
=\;& 2\ln\omega_{\mathbf{k}} + 2\sum_{n=1}^{\infty}\ln\frac{4\pi^2n^2}{\beta^2} + 2\sum_{n=1}^{\infty}\ln\left(1+\frac{\beta^2\omega_{\mathbf{k}}^2}{4\pi^2n^2}\right) \\
=\;& 2\ln\omega_{\mathbf{k}} + 2\sum_{n=1}^{\infty}\ln(4\pi^2n^2) - 4\sum_{n=1}^{\infty}\ln\beta + 2\ln\left[\prod_{n=1}^{\infty}\left(1+\frac{\beta^2\omega_{\mathbf{k}}^2}{4\pi^2n^2}\right)\right] \quad \Leftarrow \text{use } \frac{\sinh(z)}{z} = \prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2n^2}\right) \\
=\;& 2\ln\omega_{\mathbf{k}} + 2\sum_{n=1}^{\infty}\ln(4\pi^2n^2) - 4\sum_{n=1}^{\infty}\ln\beta + 2\ln\frac{2\sinh(\frac{1}{2}\beta\omega_{\mathbf{k}})}{\beta\omega_{\mathbf{k}}} \\
=\;& 2\ln\omega_{\mathbf{k}} + 2\sum_{n=1}^{\infty}\ln(4\pi^2n^2) - 4\sum_{n=1}^{\infty}\ln\beta + 2\ln(e^{\frac{1}{2}\beta\omega_{\mathbf{k}}} - e^{-\frac{1}{2}\beta\omega_{\mathbf{k}}}) - 2\ln(\beta\omega_{\mathbf{k}}) \\
=\;& -2\ln\beta + 2\sum_{n=1}^{\infty}\ln(4\pi^2n^2) - 4\sum_{n=1}^{\infty}\ln\beta + 2\ln[e^{\frac{1}{2}\beta\omega_{\mathbf{k}}}(1-e^{-\beta\omega_{\mathbf{k}}})] \\
=\;& \beta\omega_{\mathbf{k}} + 2\ln(1-e^{-\beta\omega_{\mathbf{k}}}) + 2\sum_{n=1}^{\infty}\ln(4\pi^2n^2) - 2\sum_{n=-\infty}^{\infty}\ln\beta.
\end{aligned}
$$

When substituting this expression into (1), we would obtain divergent terms, but the free energy for the Goldstone boson should not have any divergences, so the divergence should be canceled by the $\mathcal{N}'$-term in (1). After removing these terms, the free energy is given by

$$
\begin{aligned}
F &= \frac{1}{2\beta}\sum_{\mathbf{k}}[\beta\omega_{\mathbf{k}} + 2\ln(1-e^{-\beta\omega_{\mathbf{k}}})] \\
&= \frac{1}{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}} + \frac{1}{\beta}\sum_{\mathbf{k}}\ln(1-e^{-\beta\omega_{\mathbf{k}}}). \qquad (2)
\end{aligned}
$$

As a self-consistency check, you may find that (2) is just the free energy for the harmonic oscillator Hamiltonian $\hat{H} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}(\hat{b}_{\mathbf{k}}^{\dagger}\hat{b}_{\mathbf{k}} + \frac{1}{2})$ describing Goldstone bosons, i.e., the partition function is given by

$$
\begin{aligned}
Z &= \mathrm{Tr}e^{-\beta\hat{H}} \\
&= \mathrm{Tr}e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}(\hat{b}_{\mathbf{k}}^{\dagger}\hat{b}_{\mathbf{k}}+\frac{1}{2})} \\
&= \sum_{\{n_{\mathbf{k}}\}=0}^{\infty}\prod_{\mathbf{k}}e^{-\beta\omega_{\mathbf{k}}(n_{\mathbf{k}}+\frac{1}{2})} \\
&= \prod_{\mathbf{k}}\sum_{n_{\mathbf{k}}=0}^{\infty}e^{-\beta\omega_{\mathbf{k}}(n_{\mathbf{k}}+\frac{1}{2})} \\
&= \prod_{\mathbf{k}}\frac{e^{-\frac{1}{2}\beta\omega_{\mathbf{k}}}}{1-e^{-\beta\omega_{\mathbf{k}}}},
\end{aligned}
$$

and the free energy in (2) follows immediately. From the corresponding Hamiltonian formulation you may also see that the first term in (2) is the ground-state contribution and second term comes from bosonic excitations.

(b) The internal energy in the canonical ensemble is given by

$$U = \frac{1}{Z}\mathrm{Tr}(\hat{H}e^{-\beta\hat{H}}) = -\frac{\partial}{\partial\beta}\ln[\mathrm{Tr}(e^{-\beta\hat{H}})] = -\frac{\partial}{\partial\beta}\ln Z.$$

The specific heat is the the first derivative of U with respect to T,

$$\begin{aligned} C_V &= \frac{\partial U}{\partial T} \\ &= \frac{\partial}{\partial T}\left(-\frac{\partial}{\partial\beta}\ln Z\right) \\ &= \frac{\partial}{\partial T}\left(T^2\frac{\partial}{\partial T}\ln Z\right) \\ &= 2T\frac{\partial}{\partial T}\ln Z + T^2\frac{\partial^2}{\partial T^2}\ln Z \\ &= -T\left(-2\frac{\partial}{\partial T}\ln Z - T\frac{\partial^2}{\partial T^2}\ln Z\right) \\ &= -T\left[-\frac{\partial \ln Z}{\partial T} - \frac{\partial}{\partial T}\left(T\frac{\partial \ln Z}{\partial T}\right)\right] \\ &= -T\frac{\partial}{\partial T}\left(-\frac{\partial(T\ln Z)}{\partial T}\right) \\ &= -T\frac{\partial^2 F}{\partial T^2}, \end{aligned}$$

where we used $U = -\frac{\partial}{\partial\beta}\ln Z$.

By using the result in (a), the first derivative of F with respect to T is given by

$$\begin{aligned} \frac{\partial F}{\partial T} &= \sum_{\mathbf{k}}\ln(1-e^{-\beta\omega_{\mathbf{k}}}) + T\sum_{\mathbf{k}}\frac{-e^{-\beta\omega_{\mathbf{k}}}}{1-e^{-\beta\omega_{\mathbf{k}}}}\frac{\omega_{\mathbf{k}}}{T^2} \\ &= \sum_{\mathbf{k}}\ln(1-e^{-\beta\omega_{\mathbf{k}}}) - \frac{1}{T}\sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}}{e^{\beta\omega_{\mathbf{k}}}-1}. \end{aligned}$$

The second derivative is then expressed as

$$\begin{aligned} \frac{\partial^2 F}{\partial T^2} &= \sum_{\mathbf{k}}\frac{-e^{-\beta\omega_{\mathbf{k}}}}{1-e^{-\beta\omega_{\mathbf{k}}}}\frac{\omega_{\mathbf{k}}}{T^2} + \frac{1}{T^2}\sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}}{e^{\beta\omega_{\mathbf{k}}}-1} - \frac{1}{T}\sum_{\mathbf{k}}\frac{0-\omega_{\mathbf{k}}e^{\beta\omega_{\mathbf{k}}}\frac{-\omega_{\mathbf{k}}}{T^2}}{(e^{\beta\omega_{\mathbf{k}}}-1)^2} \\ &= -\frac{1}{T^3}\sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}^2}{(e^{\beta\omega_{\mathbf{k}}}-1)^2}e^{\beta\omega_{\mathbf{k}}}. \end{aligned}$$

So we have

$$C_V = -T\frac{\partial^2 F}{\partial T^2} = \frac{1}{T^2}\sum_{\mathbf{k}}\frac{e^{\omega_{\mathbf{k}}/T}}{(e^{\omega_{\mathbf{k}}/T}-1)^2}\omega_{\mathbf{k}}^2.$$

(c) Under the condition that $T \ll \max_{\mathbf{k}\in\mathrm{FBZ}}\omega_{\mathbf{k}}$, $\omega_{\mathbf{k}}$ can be well approximated by a linear dispersion relation

$$\omega_{\mathbf{k}} \simeq \sqrt{\frac{Ja^2}{C}}|\mathbf{k}|.$$

The density-of-states $D(\omega)$ is given by

$$
\begin{aligned}
D(\omega) &= \sum_{\mathbf{k}} \delta(\omega - \omega_{\mathbf{k}}) \\
&= \frac{V}{(2\pi)^3} \int_0^\infty 4\pi k^2 \delta(\omega - \omega_{\mathbf{k}}) \mathrm{d}k \\
&= \frac{V}{2\pi^2} \int_0^\infty k^2 \delta(\omega - \sqrt{\frac{Ja^2}{C}} k) \mathrm{d}k \\
&= \frac{V}{2\pi^2} \left(\frac{C}{Ja^2}\right)^{\frac{3}{2}} \int_0^\infty \left(\sqrt{\frac{Ja^2}{C}} k\right)^2 \delta\left(\omega - \sqrt{\frac{Ja^2}{C}} k\right) \mathrm{d}\left(\sqrt{\frac{Ja^2}{C}} k\right) \\
&= \frac{V}{2\pi^2} \left(\frac{C}{Ja^2}\right)^{\frac{3}{2}} \int_0^\infty \omega'^2 \delta(\omega - \omega') \mathrm{d}\omega' \\
&= \frac{V}{2\pi^2} \left(\frac{C}{Ja^2}\right)^{\frac{3}{2}} \omega^2.
\end{aligned}
$$

Then we could write down the specific heat in the integral form

$$
\begin{aligned}
C_V &= \frac{1}{T^2} \sum_{\mathbf{k}} \frac{e^{\omega_{\mathbf{k}}/T}}{(e^{\omega_{\mathbf{k}}/T} - 1)^2} \omega_{\mathbf{k}}^2 \\
&= \frac{1}{T^2} \int_0^{\omega_m} \sum_{\mathbf{k}} \delta(\omega - \omega_{\mathbf{k}}) \frac{e^{\omega_{\mathbf{k}}/T}}{(e^{\omega_{\mathbf{k}}/T} - 1)^2} \omega_{\mathbf{k}}^2 \\
&= \frac{1}{T^2} \int_0^{\omega_m} D(\omega) \frac{e^{\omega/T}}{(e^{\omega/T} - 1)^2} \omega^2 \mathrm{d}\omega \\
&= \frac{1}{T^2} \frac{V}{2\pi^2} \left(\frac{C}{Ja^2}\right)^{\frac{3}{2}} \int_0^{\omega_m} \frac{e^{\omega/T} \omega^4}{(e^{\omega/T} - 1)^2} \mathrm{d}\omega,
\end{aligned}
$$

where ω_m is the largest frequency allowed.
Define $x = \omega/T$. Since T is very small, $\frac{\omega_m}{T}$ could be regarded as $+\infty$. And we have

$$
C_V = T^3 \frac{V}{2\pi^2} \left(\frac{C}{Ja^2}\right)^{\frac{3}{2}} \int_0^\infty \frac{e^x x^4}{(e^x - 1)^2} \mathrm{d}x, \tag{3}
$$

where the integral can be worked out as follows:

$$
\begin{aligned}
\int_0^\infty \frac{e^x x^4}{(e^x - 1)^2} \mathrm{d}x &= \int_0^\infty \frac{x^4}{e^x (1 - e^{-x})^2} \mathrm{d}x \\
&= \int_0^\infty x^4 e^{-x} (1 - e^{-x})^{-2} \mathrm{d}x \\
&= \int_0^\infty x^4 e^{-x} (1 + 2e^{-x} + 3e^{-2x} + ...) \mathrm{d}x \\
&= \int_0^\infty x^4 \sum_{n=1}^\infty n e^{-nx} \mathrm{d}x \\
&= 4! \sum_{n=1}^\infty \frac{1}{n^4} \\
&= \frac{4}{15} \pi^4.
\end{aligned}
$$

Substitute this result into (3), we obtain

$$
C_V = \frac{2\pi^2}{15} V \left(\frac{C}{Ja^2}\right)^{\frac{3}{2}} T^3 \propto T^3.
$$

This agrees with Debye's T^3-law for acoustic phonons, which also have *linear* dispersion relations in the long wavelength limit.

2. **Correlation functions for $d = 2$ massless bosons** (3+2 points)

Consider the partition function of $d = 2$ massless bosonic field

$$Z = \int \mathcal{D}\phi(\mathbf{r}) \; e^{-\int d^2\mathbf{r} \frac{1}{2}[\nabla\phi(\mathbf{r})]^2},$$

which is an important theory in several research fields, such as statistical physics and string theory. During the lecture, this theory appears as the effective theory for describing the $d = 2$ classical XY model *below* the KT transition temperature.

(a) Show that the correlation function $G(\mathbf{r}_1, \mathbf{r}_2) = \langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle$ is given by

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{Z} \int \mathcal{D}\phi(\mathbf{r}) \; \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) e^{-\int d^2\mathbf{r} \frac{1}{2}[\nabla\phi(\mathbf{r})]^2} = -\frac{1}{2\pi} \ln \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{a},$$

where a is a cutoff.

(b) Show that the correlation function $\langle e^{i\alpha\phi(\mathbf{r}_1)} e^{-i\alpha\phi(\mathbf{r}_2)}\rangle$ is given by

$$\langle e^{i\alpha\phi(\mathbf{r}_1)} e^{-i\alpha\phi(\mathbf{r}_2)}\rangle = \frac{1}{Z} \int \mathcal{D}\phi(\mathbf{r}) \; e^{i\alpha\phi(\mathbf{r}_1)} e^{-i\alpha\phi(\mathbf{r}_2)} e^{-\int d^2\mathbf{r} \frac{1}{2}[\nabla\phi(\mathbf{r})]^2} = \left(\frac{a}{|\mathbf{r}_1 - \mathbf{r}_2|}\right)^{\alpha^2/2\pi}.$$

This proves that the scaling dimension of the field $e^{i\alpha\phi(\mathbf{r})}$ is $\alpha^2/4\pi$, which is essential for understanding the RG relevance/irrelevance of the cosine potential in the Sine-Gordon model.

Solution: (a) The partition function is given by

$$\begin{aligned} Z &= \int \mathcal{D}\phi(\mathbf{r}) \; e^{-\int d^2\mathbf{r} \frac{1}{2}[\nabla\phi(\mathbf{r})]^2} \\ &\propto \int \prod_{\mathbf{k}} d\phi_{\mathbf{k}} \; e^{-\frac{1}{2}\sum_{\mathbf{k}} \mathbf{k}^2 \phi(\mathbf{k})\phi(-\mathbf{k})}. \end{aligned}$$

We divide $\phi(\mathbf{k})$ into the real and imaginary parts

$$\begin{aligned} \phi(\mathbf{k}) &= \mathrm{Re}\phi(\mathbf{k}) + i\mathrm{Im}\phi(\mathbf{k}), \\ \phi(-\mathbf{k}) &= \mathrm{Re}\phi(\mathbf{k}) - i\mathrm{Im}\phi(\mathbf{k}), \end{aligned}$$

and the partition function becomes

$$Z \propto \int \prod_{\mathbf{k}:\text{half}} d\mathrm{Re}\phi(\mathbf{k}) d\mathrm{Im}\phi(\mathbf{k}) \; \exp\left\{-\sum_{\mathbf{k}:\text{half}} \mathbf{k}^2 \left[(\mathrm{Re}\phi(\mathbf{k}))^2 + (\mathrm{Im}\phi(\mathbf{k}))^2\right]\right\}.$$

Then, the correlation function in momentum space is given by

$$\begin{aligned} \langle\phi(\mathbf{k}_1)\phi(-\mathbf{k}_2)\rangle &= \frac{1}{Z} \int \prod_{\mathbf{k}:\text{half}} d\mathrm{Re}\phi(\mathbf{k}) d\mathrm{Im}\phi(\mathbf{k}) \; \phi(\mathbf{k}_1)\phi(\mathbf{k}_2) \exp\left\{-\sum_{\mathbf{k}:\text{half}} \mathbf{k}^2 \left[(\mathrm{Re}\phi(\mathbf{k}))^2 + (\mathrm{Im}\phi(\mathbf{k}))^2\right]\right\} \\ &= \frac{1}{Z} \int \prod_{\mathbf{k}:\text{half}} d\mathrm{Re}\phi(\mathbf{k}) d\mathrm{Im}\phi(\mathbf{k}) \; [\mathrm{Re}\phi(\mathbf{k}_1) + i\mathrm{Im}\phi(\mathbf{k}_1)][\mathrm{Re}\phi(\mathbf{k}_2) - i\mathrm{Im}\phi(\mathbf{k}_2)] \\ &\quad \times \exp\left\{-\sum_{\mathbf{k}:\text{half}} \mathbf{k}^2 \left[(\mathrm{Re}\phi(\mathbf{k}))^2 + (\mathrm{Im}\phi(\mathbf{k}))^2\right]\right\} \\ &= \frac{1}{Z} \int \prod_{\mathbf{k}:\text{half}} d\mathrm{Re}\phi(\mathbf{k}) d\mathrm{Im}\phi(\mathbf{k}) \; [\mathrm{Re}\phi(\mathbf{k}_1)\mathrm{Re}\phi(\mathbf{k}_2) + \mathrm{Im}\phi(\mathbf{k}_1)\mathrm{Im}\phi(\mathbf{k}_2)] \\ &\quad \times \exp\left\{-\sum_{\mathbf{k}:\text{half}} \mathbf{k}^2 \left[(\mathrm{Re}\phi(\mathbf{k}))^2 + (\mathrm{Im}\phi(\mathbf{k}))^2\right]\right\} \\ &= \delta_{\mathbf{k}_1,\mathbf{k}_2} \cdot 2\frac{1}{2\mathbf{k}_1^2} \\ &= \delta_{\mathbf{k}_1,\mathbf{k}_2} \frac{1}{\mathbf{k}_1^2}, \end{aligned}$$

where we have used $\int dx\ e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$ and $\int dx\ x^2 e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}\frac{1}{2\alpha}$.

The correlator in real space can be obtained by Fourier transformation

$$\begin{aligned} G(\mathbf{r}_1-\mathbf{r}_2) &= \langle \phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle \\ &= \frac{1}{V}\sum_{\mathbf{k}_1,\mathbf{k}_2}\langle \phi(\mathbf{k}_1)\phi(-\mathbf{k}_2)\rangle e^{i\mathbf{k}_1\cdot\mathbf{r}_1}e^{-i\mathbf{k}_2\cdot\mathbf{r}_2} \\ &= \frac{1}{V}\sum_{\mathbf{k}_1,\mathbf{k}_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\frac{1}{\mathbf{k}_1^2}e^{i\mathbf{k}_1\cdot\mathbf{r}_1}e^{-i\mathbf{k}_2\cdot\mathbf{r}_2} \\ &= \frac{1}{V}\sum_{\mathbf{k}_1}\frac{1}{\mathbf{k}_1^2}e^{i\mathbf{k}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)} \\ &\to \int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{1}{\mathbf{k}^2}e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}, \end{aligned}$$

which satisfies

$$\begin{aligned} \nabla^2_{\mathbf{r}_1}G(\mathbf{r}_1-\mathbf{r}_2) &= -\int\frac{d^2\mathbf{k}}{(2\pi)^2}e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)} \\ &= -\delta(\mathbf{r}_1-\mathbf{r}_2). \end{aligned} \tag{4}$$

Actually, this is the famous Laplace equation.

Take $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ and define $G(\mathbf{r})=G(\mathbf{r}_1-\mathbf{r}_2)$, such that $\nabla^2_{\mathbf{r}}G(\mathbf{r})=-\delta(\mathbf{r})$. Note that $G(\mathbf{r})$ only depend on $|\mathbf{r}|=|\mathbf{r}_1-\mathbf{r}_2|$, but has no angle dependence, i.e., $G(\mathbf{r})=G(r)$. We could integrate $\nabla^2_{\mathbf{r}}G(\mathbf{r})=-\delta(\mathbf{r})$ over $\mathbf{r}$ within a disk of radius r and use the Laplace operator in the polar coordinate $\nabla^2_{\mathbf{r}}=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$,

$$\begin{aligned} \text{LHS} &= \int_0^r \rho d\rho\int_0^{2\pi}d\theta\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)G(\rho) \\ &= 2\pi\int_0^r d\rho\left(\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}\right)G(\rho) \\ &= 2\pi\int_0^r d\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G(\rho)}{\partial\rho}\right) \\ &= 2\pi\left(\rho\frac{\partial G(\rho)}{\partial\rho}\right)|_0^r \\ &= 2\pi r\frac{\partial G(r)}{\partial r}, \end{aligned}$$

and

$$\begin{aligned} \text{RHS} &= \int_0^r\rho d\rho\int_0^{2\pi}d\theta[-\delta(\mathbf{r})] \qquad \Leftarrow \text{use } \delta(\mathbf{r})=\frac{1}{2\pi\rho}\delta(\rho) \\ &= -2\pi\int_0^r\rho d\rho\frac{1}{2\pi\rho}\delta(\rho) \\ &= -1. \end{aligned}$$

Therefore, we obtain

$$\frac{\partial G(r)}{\partial r}=-\frac{1}{2\pi r},$$

and immediately

$$G(r)=-\frac{1}{2\pi}\ln r+\text{const},$$

which means that

$$\begin{aligned} G(\mathbf{r}_1-\mathbf{r}_2) &= \langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle \\ &= -\frac{1}{2\pi}\ln\frac{|\mathbf{r}_1-\mathbf{r}_2|}{a}, \end{aligned}$$

where we have introduced a cutoff a.

(b) By using $\langle e^{i\alpha[\phi(\mathbf{r}_1)-\phi(\mathbf{r}_2)]}\rangle = e^{-\frac{1}{2}\alpha^2\langle[\phi(\mathbf{r}_1)-\phi(\mathbf{r}_2)]^2\rangle}$, we arrive at

$$\begin{aligned}\langle e^{i\alpha\phi(\mathbf{r}_1)}e^{-i\alpha\phi(\mathbf{r}_2)}\rangle &= e^{-\frac{1}{2}\alpha^2\langle[\phi(\mathbf{r}_1)-\phi(\mathbf{r}_2)]^2\rangle}\\ &= e^{-\frac{1}{2}\alpha^2\langle[\phi(\mathbf{r}_1)^2-2\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)+\phi(\mathbf{r}_2)^2]\rangle}\\ &= e^{-\alpha^2\langle\phi(\mathbf{r})^2\rangle}e^{\alpha^2\langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle}.\end{aligned}$$

Since $\langle\phi(\mathbf{r})^2\rangle = G(0,0)$ and $\langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle = -\frac{1}{2\pi}\ln\frac{|\mathbf{r}_1-\mathbf{r}_2|}{a}$, we could obtain

$$\begin{aligned}\langle e^{i\alpha\phi(\mathbf{r}_1)}e^{-i\alpha\phi(\mathbf{r}_2)}\rangle &= e^{-\alpha^2 G(0,0)}e^{-\frac{\alpha^2}{2\pi}\ln\frac{|\mathbf{r}_1-\mathbf{r}_2|}{a}}\\ &= e^{-\alpha^2 G(0,0)}\left(\frac{|\mathbf{r}_1-\mathbf{r}_2|}{a}\right)^{-\frac{\alpha^2}{2\pi}}\\ &\propto \left(\frac{a}{|\mathbf{r}_1-\mathbf{r}_2|}\right)^{\frac{\alpha^2}{2\pi}}.\end{aligned}$$

3. **Bogoliubov transformation** (2+3 points)

During the lecture, the Bogoliubov's theory of superfluidity has been formulated in terms of path integral. There is an equivalent formulation in terms of Hamiltonian defined with operators. The Hamiltonian describing the Bogoliubov quasiparticles is given by (c.f. page 6 in boson-2.pdf)

$$\hat{H} = \sum_{\mathbf{k}(\neq 0)}\left[\left(\frac{\mathbf{k}^2}{2m}+g\psi_0^2\right)\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \frac{1}{2}g\psi_0^2(\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}})\right],$$

where $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ are bosonic annihilation and creation operators, respectively.

To diagonalize the Hamiltonian, one needs to use the Bogoliubov transformation, which introduces a new set of bosonic operators

$$\hat{\alpha}_{\mathbf{k}} = u_{\mathbf{k}}\hat{a}_{\mathbf{k}} + v_{\mathbf{k}}\hat{a}_{-\mathbf{k}}^\dagger, \qquad \hat{\alpha}_{\mathbf{k}}^\dagger = u_{\mathbf{k}}^*\hat{a}_{\mathbf{k}}^\dagger + v_{\mathbf{k}}^*\hat{a}_{-\mathbf{k}},$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are numbers. For the present Hamiltonian, $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ can be chosen as real numbers satisfying $u_{\mathbf{k}} = u_{-\mathbf{k}}$ and $v_{\mathbf{k}} = v_{-\mathbf{k}}$.

(a) Show that $\hat{\alpha}_{\mathbf{k}}$ and $\hat{\alpha}_{\mathbf{k}}^\dagger$ satisfy the standard bosonic commutation relations $[\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'}$ if $|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2 = 1$.

(b) Find the proper choice of $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ which diagonalizes the Hamiltonian,

$$\hat{H} = \sum_{\mathbf{k}(\neq 0)}\omega_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}}^\dagger\hat{\alpha}_{\mathbf{k}} + \text{const.}$$

Check that the dispersion relation $\omega_{\mathbf{k}}$ agrees with the path integral result $\omega_{\mathbf{k}} = \sqrt{\frac{\mathbf{k}^2}{2m}\left(\frac{\mathbf{k}^2}{2m}+2g\psi_0^2\right)}$.

Solution: (a) The communtation relation is verified as follows:

$$\begin{aligned}[\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}_{\mathbf{k}'}^\dagger] &= [u_{\mathbf{k}}\hat{a}_{\mathbf{k}} + v_{\mathbf{k}}\hat{a}_{-\mathbf{k}}^\dagger, u_{\mathbf{k}'}\hat{a}_{\mathbf{k}'}^\dagger + v_{\mathbf{k}'}\hat{a}_{-\mathbf{k}'}]\\ &= u_{\mathbf{k}}u_{\mathbf{k}'}[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] + v_{\mathbf{k}}v_{\mathbf{k}'}[\hat{a}_{-\mathbf{k}}^\dagger, \hat{a}_{-\mathbf{k}'}]\\ &= (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2)\delta_{\mathbf{k}\mathbf{k}'}.\end{aligned}$$

If $|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2 = 1$ (here $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are real numbers), then

$$[\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}.$$

(b) By expressing $\hat{\alpha}_{\mathbf{k}}$ and $\hat{\alpha}_{-\mathbf{k}}^\dagger$ with $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{-\mathbf{k}}^\dagger$, we obtain

$$\begin{pmatrix}\hat{\alpha}_{\mathbf{k}}\\ \hat{\alpha}_{-\mathbf{k}}^\dagger\end{pmatrix} = \begin{pmatrix}u_{\mathbf{k}} & v_{\mathbf{k}}\\ v_{\mathbf{k}} & u_{\mathbf{k}}\end{pmatrix}\begin{pmatrix}\hat{a}_{\mathbf{k}}\\ \hat{a}_{-\mathbf{k}}^\dagger\end{pmatrix}. \tag{5}$$

We could verify easily that

$$\begin{pmatrix} u_{\mathbf{k}} & v_{\mathbf{k}} \\ v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}} \\ -v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

So that the inverse transformation of (5) is given by

$$\begin{pmatrix} \hat{a}_{\mathbf{k}} \\ \hat{a}^\dagger_{-\mathbf{k}} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}} \\ -v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \hat{\alpha}_{\mathbf{k}} \\ \hat{\alpha}^\dagger_{-\mathbf{k}} \end{pmatrix}.$$

Let $A_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} + g\psi_0^2$ and $B_{\mathbf{k}} = \frac{1}{2}g\psi_0^2$ (in the present problem $B_{\mathbf{k}}$ does not have momentum dependence). The Hamiltonian can now be written in terms of Bogoliubov quasiparticle operators as

$$\begin{aligned} \hat{H} &= \sum_{\mathbf{k}(\neq 0)} A_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + B_{\mathbf{k}} (\hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}} + \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}}) \\ &= \sum_{\mathbf{k}(\neq 0)} A_{\mathbf{k}} (u_{\mathbf{k}} \hat{\alpha}^\dagger_{\mathbf{k}} - v_{\mathbf{k}} \hat{\alpha}_{-\mathbf{k}})(u_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}} - v_{\mathbf{k}} \hat{\alpha}^\dagger_{-\mathbf{k}}) + B_{\mathbf{k}} [(u_{\mathbf{k}} \hat{\alpha}^\dagger_{\mathbf{k}} - v_{\mathbf{k}} \hat{\alpha}_{-\mathbf{k}})(u_{\mathbf{k}} \hat{\alpha}^\dagger_{-\mathbf{k}} - v_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}}) \\ &\quad + (u_{\mathbf{k}} \hat{\alpha}_{-\mathbf{k}} - v_{\mathbf{k}} \hat{\alpha}^\dagger_{\mathbf{k}})(u_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}} - v_{\mathbf{k}} \hat{\alpha}^\dagger_{-\mathbf{k}})] \\ &= \sum_{\mathbf{k}(\neq 0)} [A_{\mathbf{k}} (u^2_{\mathbf{k}} + v^2_{\mathbf{k}}) - 4 B_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}] \hat{\alpha}^\dagger_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}} + [B_{\mathbf{k}} (u^2_{\mathbf{k}} + v^2_{\mathbf{k}}) - A_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}](\hat{\alpha}^\dagger_{\mathbf{k}} \hat{\alpha}^\dagger_{-\mathbf{k}} + \hat{\alpha}_{\mathbf{k}} \hat{\alpha}_{-\mathbf{k}}) + \text{const.} \end{aligned}$$

To diagonalize H, the off-diagonal term is necessarily zero, i.e.,

$$B_{\mathbf{k}} (u^2_{\mathbf{k}} + v^2_{\mathbf{k}}) - A_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} = 0.$$

Together with the constraint $u^2_{\mathbf{k}} - v^2_{\mathbf{k}} = 1$, we obtain

$$u_{\mathbf{k}} = \sqrt{\frac{1}{2}(\Delta_{\mathbf{k}} + 1)}, \qquad v_{\mathbf{k}} = \sqrt{\frac{1}{2}(\Delta_{\mathbf{k}} - 1)},$$

where $\Delta_{\mathbf{k}} = \sqrt{\frac{A^2_{\mathbf{k}}}{A^2_{\mathbf{k}} - 4B^2_{\mathbf{k}}}}$.

After substituting the coherent factors $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ into $\hat{H}$, we obtain

$$\begin{aligned} \hat{H} &= \sum_{\mathbf{k}(\neq 0)} [A_{\mathbf{k}} (u^2_{\mathbf{k}} + v^2_{\mathbf{k}}) - 4 B_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}] \hat{\alpha}^\dagger_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}} + \text{const} \\ &= \sum_{\mathbf{k}(\neq 0)} \left(A_{\mathbf{k}} \Delta_{\mathbf{k}} - 2 B_{\mathbf{k}} \sqrt{\Delta^2_{\mathbf{k}} - 1} \right) \hat{\alpha}^\dagger_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}} + \text{const} \\ &= \sum_{\mathbf{k}(\neq 0)} \sqrt{A^2_{\mathbf{k}} - 4 B^2_{\mathbf{k}}} \hat{\alpha}^\dagger_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}} + \text{const} \\ &= \sum_{\mathbf{k}(\neq 0)} \sqrt{\frac{\mathbf{k}^2}{2m} \left(\frac{\mathbf{k}^2}{2m} + 2g\psi_0^2 \right)} \hat{\alpha}^\dagger_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}} + \text{const}, \end{aligned}$$

from which we can read out the dispersion relation

$$\omega_{\mathbf{k}} = \sqrt{\frac{\mathbf{k}^2}{2m} \left(\frac{\mathbf{k}^2}{2m} + 2g\psi_0^2 \right)}.$$