

## §4. Physics in $d=1$

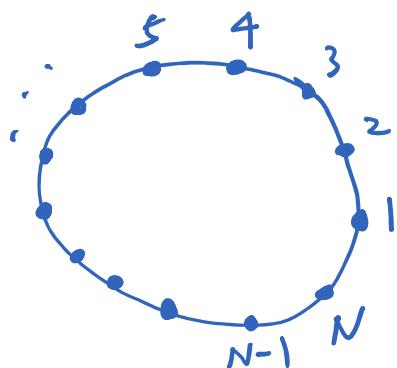
\*) Effective theory for  $d=1$  antiferromagnetic (AFM)

Heisenberg models ("Haldane's conjecture")

Nobel Prize 2016

$$\hat{H} = J \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_{j+1}$$

$J > 0$ , even  $N$ ,  
lattice spacing  $a$



$$\vec{s}_j^2 = S(S+1)$$

$$\mathcal{Z} = \int d\vec{\tau} e^{-A[\{\vec{\tau}\}]}$$

$$A[\{\vec{\tau}\}] = \int_0^t d\tau \left[ iS \sum_{j=1}^N (1 - \cos \theta_j) \dot{\phi}_j + JS^2 \sum_{j < l} \vec{\tau}_j \cdot \vec{\tau}_l \right]$$

$$\vec{\tau}_j = (\sin \theta_j \cos \phi_j, \sin \theta_j \sin \phi_j, \cos \theta_j)$$

Static mean-field ansatz + Gaussian fluctuations

would fail even at  $T=0$ !

( infrared divergences  $\Rightarrow$  Mermin-Wagner theorem )

Nevertheless, dominant configurations:

$$\dots \uparrow \downarrow \nearrow \downarrow \nearrow \downarrow \uparrow \swarrow \dots$$

(close to Néel order)

$$\vec{s}_j = (-1)^j \vec{n}_j + \frac{q}{S} \vec{L}_j$$

↑                            ↙

slowly varying part      rapidly varying part

$|\vec{n}_j| = 1$

(For  $d \geq 2$ ,  $\vec{n}_j$  is the order-parameter field.)

Goal: develop an effective theory describing  
the dynamics of slowly varying variables.

(Spirit of effective theory: integrate over  
rapidly varying, high energy variables)

$$\begin{aligned}
 |\vec{s}_j|^2 &= |(-1)^j \vec{n}_j + \frac{a}{s} \vec{l}_j|^2 \\
 &= 1 + \frac{2a}{s} (-1)^j \underbrace{\vec{n}_j \cdot \vec{l}_j}_{\parallel 0} + \left(\frac{a}{s}\right)^2 |\vec{l}_j|^2 \\
 &= 1 + O(a^2)
 \end{aligned}$$

constraint :  $\underbrace{\vec{n}_j \cdot \vec{l}_j = 0}$

$$\begin{aligned}
 H(\{\vec{s}_j\}) &= JS^2 \sum_{j \neq l} \vec{s}_j \cdot \vec{s}_l \\
 &= JS^2 \sum_{j=1}^N \left[ (-1)^j \vec{n}_j + \frac{a}{s} \vec{l}_j \right] \cdot \left[ \underbrace{(-1)^{j+1} \vec{n}_{j+1}}_{-(-1)^j} + \frac{a}{s} \vec{l}_{j+1} \right] \\
 &= -JS^2 \sum_{j=1}^N \vec{n}_j \cdot \vec{n}_{j+1} \\
 &\quad + JSa \sum_{j=1}^N (-1)^j (\underbrace{\vec{n}_j \cdot \vec{l}_{j+1} - \vec{l}_j \cdot \vec{n}_{j+1}}_{\downarrow} \\
 &\quad + \vec{n}_j \cdot \vec{l}_{j-1}) \\
 &\quad + Ja^2 \sum_{j=1}^N \vec{l}_j \cdot \vec{l}_{j+1}
 \end{aligned}$$

$$\xrightarrow{\text{continuum}} -JS^2 \sum_{j=1}^N \vec{n}_j \cdot [\vec{n}_j + a \vec{n}_j \cdot \partial_x \vec{n}_j + \frac{1}{2} a^2 \vec{n}_j \cdot \partial_x^2 \vec{n}_j + \dots]$$

$$\stackrel{||}{=} 0 \text{ because } 1 = |\vec{n}_j|^2 \\ \text{and } 0 = 2 \vec{n}_j \cdot \partial_x \vec{n}_j$$

$$+ JSa \sum_{j=1}^N (-1)^j \left[ \underbrace{\vec{n}_j \cdot \vec{L}_j}_{=0} + a \vec{n}_j \cdot \cancel{\partial_x \vec{L}_j} + \dots \right. \\ \left. + \underbrace{\vec{n}_j \cdot \vec{L}_j}_{=0} - a \vec{n}_j \cdot \cancel{\partial_x \vec{L}_j} + \dots \right]$$

$$+ Ja^2 \sum_{j=1}^N |\vec{L}_j|^2 + \dots$$

$$= -JS^2 N - \frac{1}{2} JS^2 a^2 \sum_{j=1}^N \vec{n}_j \cdot \partial_x^2 \vec{n}_j$$

classical ground-state energy

$$\cancel{\partial_x(\vec{n}_j \cdot \partial_x \vec{n}_j)} - \cancel{(\partial_x \vec{n}_j)^2}$$

total derivative

$$+ Ja^2 \sum_{j=1}^N |\vec{L}_j|^2 + O(a^3)$$

$$\simeq \text{const.} + \int_0^L dx \left[ \frac{1}{2} JS^2 a^2 (\partial_x \vec{n})^2 + Ja |\vec{L}|^2 \right]$$

Berry's phase:

$$\vec{R}_j = (-1)^j \vec{n}_j + \frac{a}{S} \vec{L}_j$$

$$iS \sum_{j=1}^N \omega(\{\vec{R}_j\})$$

$$\omega(\vec{R}) = \int_0^\beta d\tau (1 - \cos \theta) \dot{\varphi}$$

$$= iS \sum_j \left[ \omega\left(\{(-1)^j \vec{n}_j\}\right) \right.$$

$$+ \int_0^\beta d\tau \underbrace{\frac{\delta \omega(\{\vec{R}_j(\tau)\})}{\delta \vec{R}_j(\tau)}}_{\parallel} \Big|_{\vec{R}_j = (-1)^j \vec{n}_j} \cdot \frac{a}{S} \vec{L}_j + \dots \Big]$$

$$\partial_\tau \vec{R}_j \times \vec{R}_j \quad (\text{see below})$$

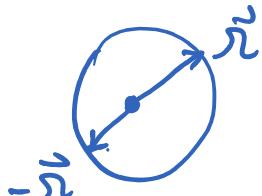
Note that

$$-\vec{R} = (\sin(\pi - \theta) \cos(\pi + \varphi), \sin(\pi - \theta) \sin(\pi + \varphi), \cos(\pi + \varphi))$$

$$\theta \rightarrow \pi - \theta$$

$$\varphi \rightarrow \pi + \varphi$$

$$\Rightarrow \omega(-\vec{R}) = \int_0^\beta d\tau (1 + \cos \theta) \dot{\varphi}$$



$$= - \int_0^\beta d\tau (1 - \cos \theta) \dot{\varphi} + \underbrace{2 \int_0^\beta d\tau \dot{\varphi}}_{\parallel} \\ = - \omega(\vec{R}) \quad 2[\varphi(\beta) - \varphi(0)] = 0$$

single spin:  $\omega(\vec{r}) = \int_0^{\beta} d\tau (1 - \omega_0 \theta) \dot{\varphi}$

Variation:  $\delta\omega = \int_0^{\beta} d\tau \sin\theta \delta\theta \dot{\varphi} + (1 - \omega_0 \theta) \delta\dot{\varphi}$

$$\cancel{\partial_\tau [(1 - \omega_0 \theta) \delta\dot{\varphi}]} - \sin\theta \cdot \dot{\theta} \delta\dot{\varphi}$$

$$= \int_0^{\beta} d\tau (\dot{\varphi} \delta\theta - \dot{\theta} \delta p) \sin\theta$$

$$= \int_0^{\beta} d\tau \delta\vec{r} \cdot (\partial_\tau \vec{r} \times \vec{r})$$

$$\Rightarrow \underbrace{\frac{\delta\omega}{\delta\vec{r}}}_{\text{green wavy line}} = \partial_\tau \vec{r} \times \vec{r}$$

$$\Rightarrow iS \sum_{j=1}^N \omega(\{\vec{r}_j\})$$

$$= iS \sum_j \left[ \underbrace{(-1)^j}_{\text{blue wavy line}} \omega(\{\vec{n}_j\}) + \int_0^{\beta} d\tau (\partial_\tau \vec{n}_j \times \vec{n}_j) \cdot \frac{a}{S} \vec{L}_j + \dots \right]$$

$$\simeq iS \sum_j (-1)^j \omega(\{\vec{n}_j\})$$

$$+ i \int_0^{\beta} d\tau \int_0^L dx (\partial_\tau \vec{n} \times \vec{n}) \cdot \vec{L}$$

$$\Rightarrow Z = \underbrace{\int D\vec{n} D\vec{L} \delta(\vec{n} \cdot \vec{L})}_{\downarrow} e^{-A[\vec{n}, \vec{L}]}$$

ensure  $\vec{n} \cdot \vec{L} = 0$

$$A[\vec{n}, \vec{L}] = iS \sum_j (-1)^j w(\{\vec{n}_j\})$$

$$+ \int_0^B d\tau \int_0^L dx \left[ \underbrace{i(\partial_\tau \vec{n} \times \vec{n}) \cdot \vec{L}}_{+ \frac{1}{2} JS a (\partial_x \vec{n})^2 + Ja |\vec{L}|^2} \right]$$

complete square!

$$Ja \left[ \vec{L} + \underbrace{\frac{i}{2Ja} (\partial_\tau \vec{n} \times \vec{n})}_{\downarrow} \right]^2$$

orthogonal to  $\vec{n}$ ,

constraint satisfied and

Gaussian integration possible!

Integrate over  $\vec{L}$ :

$$A[\vec{n}] = iS \sum_j (-1)^j w(\{\vec{n}_j\})$$

$$+ \int_0^B d\tau \int_0^L dx \left[ \frac{1}{2} JS^2 a^2 (\partial_x \vec{n})^2 + \frac{1}{4Ja} (\partial_\tau \vec{n})^2 \right]$$

rescale:  $x_0 = c\tau$

$$x_1 = x$$

$$\xrightarrow{\quad} \text{velocity } c = \sqrt{\sum J S a} \quad iS \sum_j (-1)^j w(\{\vec{n}_j\}) + \frac{1}{g} \int_0^B dx_0 \int_0^L dx_1 (\partial_{\mu} \vec{n})^2$$

$$g = \frac{\sqrt{\sum J S a}}{S}$$

$$iS \sum_j (-1)^j \omega(\{n_j\})$$

$$= i \sum_j^S [ \omega(\{n_{2j}\}) - \omega(\{n_{2j-1}\}) ]$$

continuum  $\rightarrow i \sum_j^S \int_0^\beta d\tau \int_0^L dx \frac{\delta \omega(\{\vec{n}\})}{\delta \vec{n}} \cdot \frac{\partial \vec{n}}{\partial x}$

  
 $\frac{\partial \vec{n}}{\partial \tau} \parallel \vec{n} \times \vec{n}$   
 $\parallel$   
 $\frac{\partial \vec{n}}{\partial x} \parallel \vec{n}$

$$= -i \sum_j^S \int_0^\beta d\tau \int_0^L dx \vec{n} \cdot (\partial_\tau \vec{n} \times \partial_x \vec{n})$$

rescale

$$= -i \sum_j^S \int_0^{fc} dx_0 \int_0^L dx_1 \vec{n} \cdot (\partial_{x_0} \vec{n} \times \partial_{x_1} \vec{n})$$

$$= -i 2\pi S Q(\vec{n}),$$

where  $Q(\vec{n}) = \frac{1}{4\pi} \int_0^{fc} dx_0 \int_0^L dx_1 \vec{n} \cdot (\partial_{x_0} \vec{n} \times \partial_{x_1} \vec{n})$

  
= integer      topological invariant !

$$\Rightarrow z = \int d\vec{n} e^{-i 2\pi S Q(\vec{n})} e^{-A[\vec{n}]}$$

  
 unit vector

$$A[\vec{n}] = \frac{1}{g} \int_0^{fc} dx_0 \int_0^L dx_1 (\partial_\mu \vec{n})^2$$

- Integer  $S$ :

$$e^{-iz\pi S Q(\vec{n})} = +1$$

$$Z = \int D\vec{n} e^{-A[\vec{n}]}$$

Standard  $O(3)$  non-linear sigma model (NLSM), which has been studied in field theory community.

It has a mass gap and exponentially decaying correlations.

- Half-integer  $S$

$$e^{-iz\pi S Q(\vec{n})} = (-1)^{Q(\vec{n})}$$

$$Z \simeq \int D\vec{n} \underbrace{(-1)^{Q(\vec{n})}}_{\text{topological term}} e^{-A[\vec{n}]}$$

$O(3)$  NLSM + topological term

No reliable analytical / numerical approaches available for understanding its physics ...

The interference due to the sign factor is expected to be important!

Luckily, the Bethe ansatz solution for  $S=1/2$  AFM Heisenberg chain is available.

This indicates gapless excitations for half-integer  $S$ !

Haldane's conjecture:

$d=1$  AFM Heisenberg chains have

- { a gap and exponentially decaying correlations for integer  $S$
- no gap for half-integer  $S$