

§4. Physics in $d=1$

*) Bethe Ansatz

spin- $1/2$ AFM Heisenberg chain

$$\hat{H} = \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_{j+1}$$

First solution by Hans Bethe in 1931
 (paper written in German), now known as
 "Coordinate Bethe Ansatz".

The main steps can be found in Fradkin's
 and Nagaosa's books.

Here we will instead use the so-called
 "Algebraic Bethe Ansatz" (also known as
 "Quantum inverse-scattering method") by
 the "Leningrad school" (leading researcher:
 L.D. Faddeev) in 1970s ~ 1980s.

Reference: L.D. Faddeev, arxiv, hep-th/9605187

- Main ideas:
- 1) Find/construct a set of mutually commuting operators (Hamiltonian is one of them).
 - 2) Use powerful algebraic approach to find eigenenergies/eigenvectors.

Lax operator:

number, usually called "spectral parameter"

$$L_{n,a}(u) = u \mathbb{1}_n \otimes \mathbb{1}_a + i \sum_{\nu=x,y,z} S_n^\nu \otimes \sigma_a^\nu$$

physical spin-1/2 auxiliary spin-1/2 spin-1/2 operators

$$= \begin{pmatrix} u + iS_n^z & iS_n^- \\ iS_n^+ & u - iS_n^z \end{pmatrix}_a$$

$$L_{n,a}(u) = \begin{array}{c} n \\ \square \\ a \end{array}$$

Example:

$$\langle \uparrow | L_{n,a}(u) | \downarrow \rangle_a = \langle \uparrow | - \begin{array}{c} \square \\ u \end{array} | \downarrow \rangle_a = iS_n^-$$

Permutation operator :

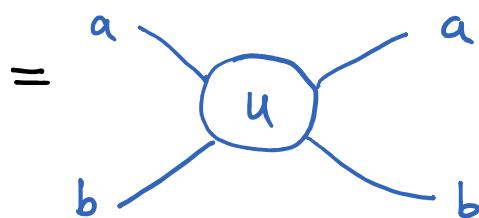
$$P_{n,a} = \frac{1}{2} (\mathbb{1}_n \otimes \mathbb{1}_a + 2 \sum_{\nu=x,y,z} S_n^\nu \otimes \sigma_a^\nu)$$

$$= \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}_{n,a}$$

$$P_{n,a} |\alpha\rangle_n \otimes |\beta\rangle_a = |\beta\rangle_n \otimes |\alpha\rangle_a$$

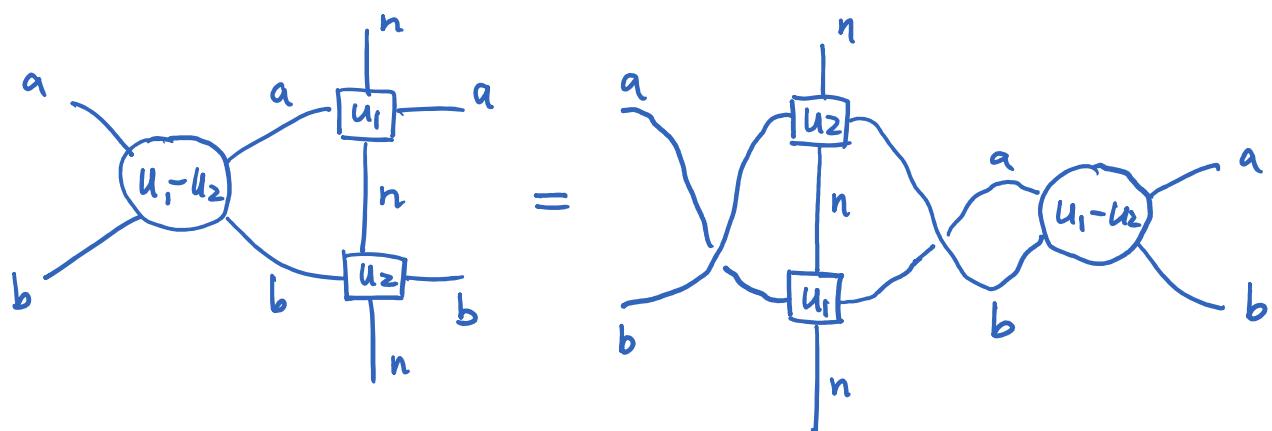

R-matrix :

$$R_{ab}(u) = u \mathbb{1}_a \otimes \mathbb{1}_b + i P_{a,b}$$



Yang-Baxter equation (YBE)

$$R_{ab}(u_1 - u_2) L_{n,a}(u_1) L_{n,b}(u_2) = L_{n,b}(u_2) L_{n,a}(u_1) R_{ab}(u_1 - u_2)$$



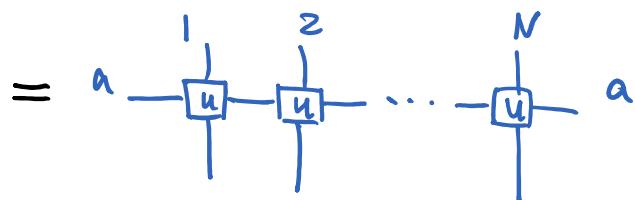
YBE is the basic building block for commuting operators!

Monodromy (defined with $\underbrace{N+1}_{\uparrow}$ spin- $1/2$'s):

N physical + 1 auxiliary

$$T_a(u) = L_{N,a}(u) L_{N-1,a}(u) \cdots L_{1,a}(u)$$

$$= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a$$



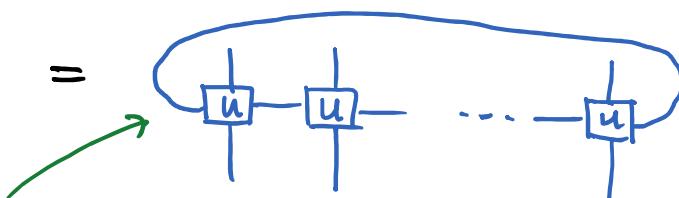
Example: $B(u) = \langle \uparrow | \underset{a}{\dots} | \uparrow \rangle_a$

\uparrow
operator acting on N physical spin- $1/2$'s

Transfer matrix:

$$t(u) = \text{tr}_a T_a(u) \xrightarrow{A(u)} \xrightarrow{D(u)}$$

$$= \underbrace{\langle \uparrow | T_a(u) | \uparrow \rangle_a}_{\text{A}} + \underbrace{\langle \downarrow | T_a(u) | \downarrow \rangle_a}_{\text{D}}$$



"matrix-product operator" (MPO) in tensor network language.

Now comes the amazing result:

$$\underline{[t(u_1), t(u_2)] = 0 \quad \forall u_1, u_2]}$$

This is amazing because we have now a large number of mutually commuting operators!

$$\begin{aligned} t(u) &= \text{tr}_a \left[\begin{pmatrix} u + iS_N^z & iS_N^- \\ iS_N^+ & u - iS_N^z \end{pmatrix}_a \begin{pmatrix} u + iS_{N-1}^z & iS_{N-1}^- \\ iS_{N-1}^+ & u - iS_{N-1}^z \end{pmatrix}_a \right. \\ &\quad \dots \left. \begin{pmatrix} u + iS_1^z & iS_1^- \\ iS_1^+ & u - iS_1^z \end{pmatrix}_a \right] \\ &= \text{tr}_a \left[\begin{pmatrix} u^2 + iu(S_{N-1}^z + S_N^z) - S_{N-1}^z S_N^z - S_{N-1}^+ S_N^- & \dots \\ \dots & u^2 - iu(S_{N-1}^z + S_N^z) - S_{N-1}^z S_N^z - S_{N-1}^- S_N^+ \end{pmatrix} \dots \right] \\ &= \sum_{p=0}^N u^p \hat{O}_p \quad \leftarrow \text{polynomial of } u \text{ with operators being the "coefficients".} \end{aligned}$$

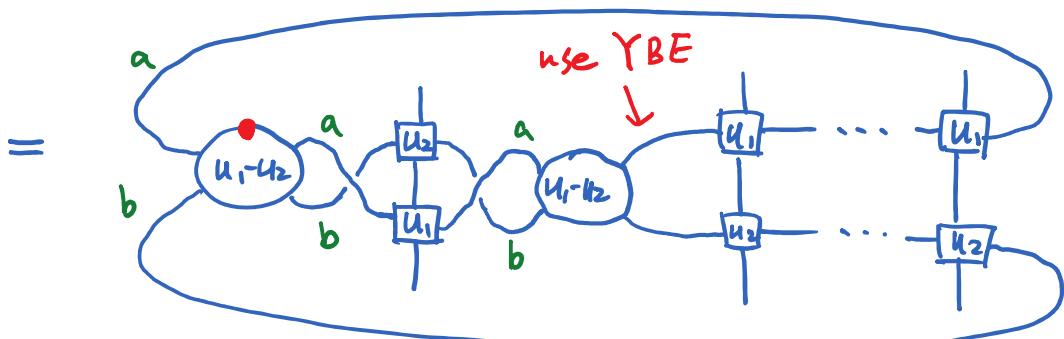
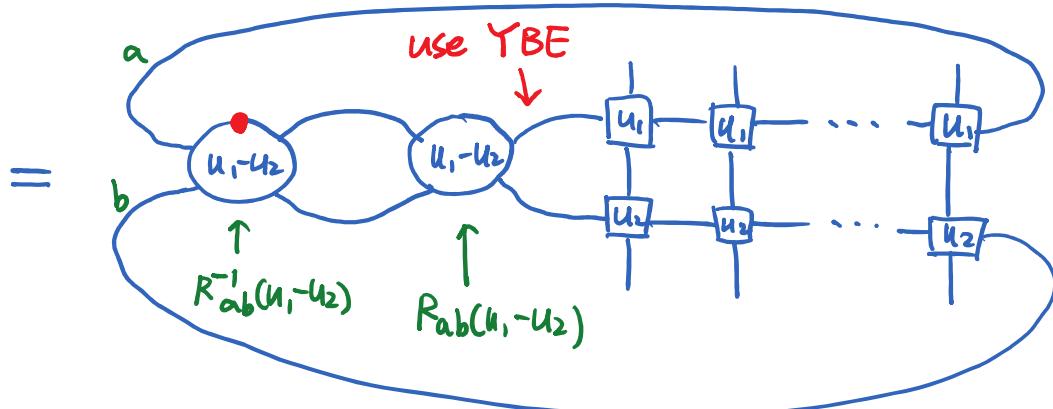
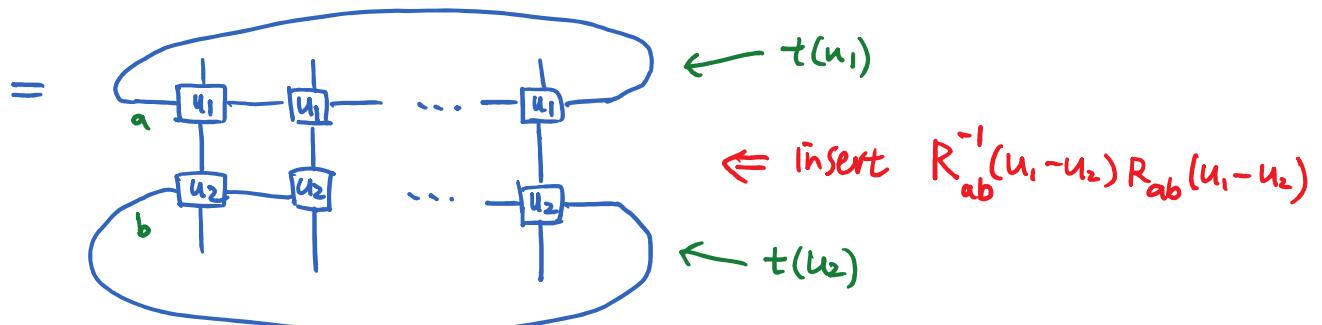
$$[t(u_1), t(u_2)] = 0 \quad \forall u_1, u_2$$



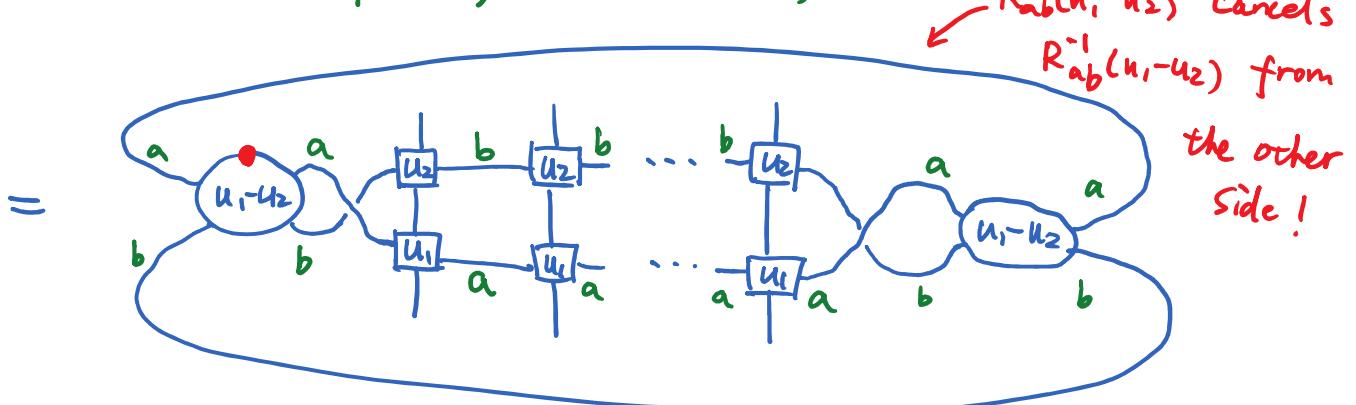
$$[\hat{O}_p, \hat{O}_q] = 0 \quad \forall p, q = 0, \dots, N$$

Proof of $[t(u_1), t(u_2)] = 0$:

$t(u_1) t(u_2)$



= ... (repeatedly use YBE)



$$\begin{aligned}
 &= \text{Diagram showing two horizontal rows of boxes labeled } u_2 \text{ and } u_1. \\
 &\quad \text{The top row has three boxes, each with a green label } b \text{ above it.} \\
 &\quad \text{The bottom row has three boxes, each with a green label } a \text{ below it.} \\
 &\quad \text{Vertical connections exist between adjacent } u_2 \text{ boxes and between adjacent } u_1 \text{ boxes.} \\
 &\quad \text{Horizontal connections exist between } u_2 \text{ and } u_1 \text{ boxes at the same height.} \\
 \\
 &= t(u_2) t(u_1) \quad \text{Q.E.D.}
 \end{aligned}$$

Message: If a solution of the Yang-Baxter equation is found, a set of mutually commuting operators can be constructed.

Example 1: translation operator \hat{T}

$$\begin{aligned}
 L_{n,a}(u) &= u \mathbb{1}_n \otimes \mathbb{1}_a + i \sum_{v=x,y,z} S_n^v \otimes \sigma_a^v \\
 &= \left(u - \frac{i}{2}\right) \mathbb{1}_n \otimes \mathbb{1}_a + i P_{n,a}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow t(u = \frac{i}{2}) &= \text{tr}_a [L_{N,a}(u = \frac{i}{2}) L_{N-1,a}(u = \frac{i}{2}) \cdots L_{1,a}(u = \frac{i}{2})] \\
 &= i^N \text{tr}_a (P_{N,a} P_{N-1,a} \cdots P_{1,a}) \\
 (\text{Proof: exercise}) &\quad \Rightarrow i^N \underline{\text{tr}_a} (P_{1,2} P_{2,3} \cdots P_{N-1,N} \underline{P_{N,a}}) \\
 &= i^N P_{1,2} P_{2,3} \cdots P_{N-1,N} \quad \frac{1}{2} (\mathbb{1}_N \otimes \mathbb{1}_a + \sigma_N^v \otimes \sigma_a^v)
 \end{aligned}$$

This is just the translation operator \hat{T} :

$$\begin{aligned}
 & P_{12} P_{23} \cdots \underbrace{P_{N-1,N}}_{\text{---}} |S_1, S_2, \dots, \underline{S_{N-1}, S_N}\rangle \\
 = & P_{12} P_{23} \cdots \underbrace{P_{N-2,N-1}}_{\text{---}} |S_1, S_2, \dots, \underline{S_{N-2}, S_N, S_{N-1}}\rangle \\
 = & \dots \\
 = & |S_N, S_1, S_2, \dots, \underline{S_{N-2}, S_{N-1}}\rangle \\
 \Rightarrow \hat{T} = & \underbrace{P_{12} P_{23} \cdots P_{N-1,N}}_{\text{---}} \quad \text{with} \quad \hat{T} \hat{S}_j^v \hat{T}^{-1} = \hat{S}_{j+1}^v
 \end{aligned}$$

Example 2: Hamiltonian \hat{H}

$$\begin{aligned}
 \frac{d}{du} \ln t(u) \Big|_{u=\frac{i}{2}} &= \left[\frac{d}{du} t(u) \cdot t(u)^{-1} \right] \Big|_{u=\frac{i}{2}}
 \end{aligned}$$

Order does NOT matter,
since they commute anyway.

$$\begin{aligned}
 \frac{d}{du} t(u) \Big|_{u=\frac{i}{2}} &= \frac{d}{du} \text{tra} [L_{N,a}(u) L_{N-1,a}(u) \cdots L_{1,a}(u)]
 \end{aligned}$$
$$\begin{aligned}
 \frac{d}{du} L_{n,a}(u) &= \frac{d}{du} [(u - \frac{i}{2}) \mathbb{1}_n \otimes \mathbb{1}_a + i P_{n,a}] \\
 &= \mathbb{1}_n \otimes \mathbb{1}_a
 \end{aligned}$$

$$\begin{aligned}
 \left. \frac{d}{du} t(u) \right|_{u=\frac{i}{2}} &= \text{tr}_a \left[(\mathbb{1}_N \otimes \mathbb{1}_a) \cdot i P_{N-1,a} \cdots i P_{1,a} \right] \\
 &\quad + \cdots \left(\frac{d}{du} \text{ acting on other Lax operators} \right) \\
 &= i^{N-1} \text{tr}_a \left[\sum_{j=1}^N P_{N,a} \cdots \hat{P}_{j,a} \cdots P_{1,a} \right] \\
 &\quad \uparrow \\
 &\text{This means } P_{j,a} \text{ is absent!} \\
 &= i^{N-1} \sum_{j=1}^N \text{tr}_a \left(P_{12} P_{23} \cdots P_{j-1,j+1} \cdots P_{N-1,N} P_{N,a} \right) \\
 &= i^{N-1} \sum_{j=1}^N P_{12} P_{23} \cdots P_{j-1,j+1} \cdots P_{N-1,N}
 \end{aligned}$$

$$\begin{aligned}
 &\left[\frac{d}{du} t(u) \cdot t(u)^{-1} \right] \Big|_{u=\frac{i}{2}} \xrightarrow{t(u=\frac{i}{2})^{-1} = i^N \hat{\tau}} \\
 &= i^{N-1} \sum_{j=1}^N P_{12} P_{23} \cdots P_{j-1,j+1} \cdots P_{N-1,N} \times (i^N P_{12} P_{23} \cdots P_{N-1,N})^{-1} \\
 &= -i \sum_{j=1}^N P_{12} P_{23} \cdots P_{j-1,j+1} \cdots P_{N-1,N} \times (P_{N-1,N} P_{N-2,N-1} \cdots P_{12}) \\
 &\text{use } P_{n,a} P_{ab} P_{n,b} = P_{n,b} \\
 &\downarrow \\
 &= -i (P_{12} + P_{23} + \cdots + P_{N-1,N}) \\
 &= -i \sum_{j=1}^N \left(\frac{1}{2} + 2 \vec{s}_j \cdot \vec{s}_{j+1} \right) \quad \xleftarrow{\quad} \quad P_{j,j+1} = \frac{1}{2} + 2 \vec{s}_j \cdot \vec{s}_{j+1}
 \end{aligned}$$

Thus, we obtain

$$\left. i \frac{d}{du} \ln t(u) \right|_{u=\frac{i}{2}} = \sum_{j=1}^N \left(2 \vec{s}_j \cdot \vec{s}_{j+1} + \frac{1}{z} \right)$$

This is just the spin-1/2 AFM Heisenberg chain
(up to overall constant and factor).

Summary: The transfer matrix $t(u)$ constructed
from Lax operators satisfying the
Yang-Baxter equation forms a set of
mutually commuting operators.

(# of these operators $\sim N$)
 \hookrightarrow integrability !

These operators include translation operator \hat{T} ,
Hamiltonian \hat{H} , ...
(They share common eigenvectors with $t(u)$!)

Next step: diagonalize $t(u)$.