

§4. Physics in $d=1$

* Bethe Ansatz (cont'd)

- Diagonalize the transfer matrix $t(u)$

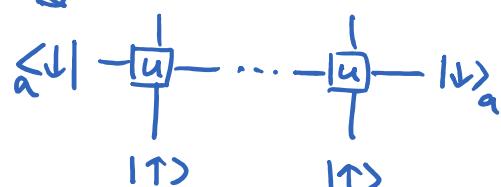
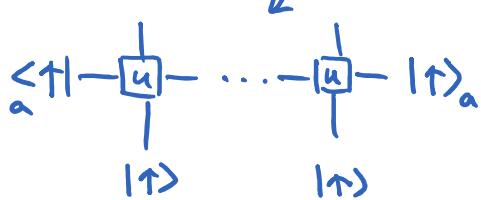
$$\begin{aligned} t(u) &= \text{tra } T_a(u) \\ &= \text{tra}_a \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a \\ &= A(u) + D(u) \end{aligned}$$

Reference state: $|0\rangle \equiv |\uparrow\uparrow\cdots\uparrow\rangle_{1\rightarrow N}$

This is obviously an eigenstate of \hat{T} and \hat{H} ,
so it must be an eigenvector of $t(u)$.

Let's work out its eigenvalue:

$$t(u)|0\rangle = \underbrace{A(u)|0\rangle}_{\downarrow} + \underbrace{D(u)|0\rangle}_{\downarrow}$$



Use $L_{n,a} = (u - \frac{i}{2}) \mathbb{1}_n \otimes \mathbb{1}_a + i P_{n,a}$

$$\begin{aligned}
 A(u) |0\rangle &= \underbrace{\langle \uparrow|}_{a} L_{N,a}(u) \cdots \underbrace{L_{1,a}(u)}_{a} |\uparrow\rangle_a \otimes |\uparrow\uparrow\cdots\uparrow\rangle_{1\rightarrow N} \\
 &= \underbrace{\langle \uparrow|}_{a} L_{N,a}(u) \cdots \underbrace{L_{2,a}(u)}_{a} [(u - \frac{i}{2}) + i] |\uparrow\rangle_a \otimes |\uparrow\uparrow\cdots\uparrow\rangle_{1\rightarrow N} \\
 &= \dots \\
 &= [(u - \frac{i}{2}) + i]^N \underbrace{\langle \uparrow|}_{a} |\uparrow\rangle_a \otimes |\uparrow\uparrow\cdots\uparrow\rangle_{1\rightarrow N} \\
 &= (u + \frac{i}{2})^N |0\rangle
 \end{aligned}$$

$$\begin{aligned}
 D(u) |0\rangle &= \underbrace{\langle \downarrow|}_{a} L_{N,a}(u) \cdots \underbrace{L_{1,a}(u)}_{a} |\downarrow\rangle_a \otimes |\uparrow\uparrow\cdots\uparrow\rangle_{1\rightarrow N} \\
 &= \underbrace{\langle \downarrow|}_{a} L_{N,a}(u) \cdots \underbrace{L_{2,a}(u)}_{a} [(u - \frac{i}{2}) |\downarrow\rangle_a \otimes |\uparrow\uparrow\cdots\uparrow\rangle_{1\rightarrow N} \\
 &= \dots \\
 &= (u - \frac{i}{2})^N |0\rangle
 \end{aligned}$$

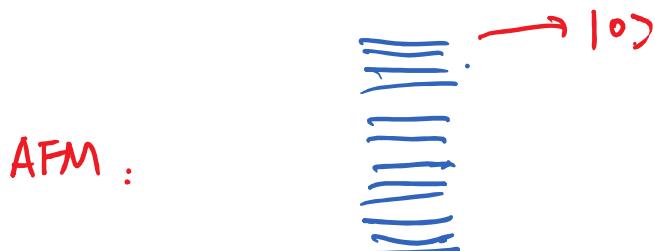
No contribution, because
 there is NO chance to permute
 ↓-spin back to the auxiliary
 space a !

$$\Rightarrow t(u) |0\rangle = \left[(u + \frac{i}{2})^N + (u - \frac{i}{2})^N \right] |0\rangle$$

\downarrow
 eigenvector ✓

If \hat{H} has FM interactions, $|0\rangle$ is already the ground state.

However, if \hat{H} has AFM interactions, $|0\rangle$ is the highest energy state!



We need to determine ground / low-energy states.

Idea: Find "lowering" operator, like \hat{s}^- in $SU(2)$

$$|s,s\rangle \rightarrow \hat{s}^- |s,s\rangle \rightarrow \dots \rightarrow (\hat{s}^-)^{2s} |s,s\rangle$$

\nearrow
 $|s,s-1\rangle$ \searrow
 $|s,-s\rangle$

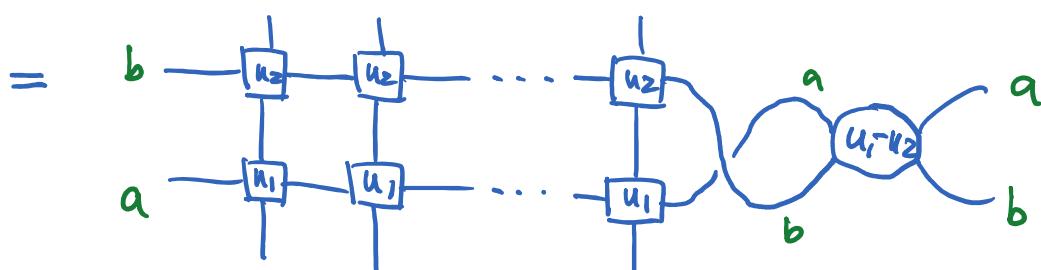
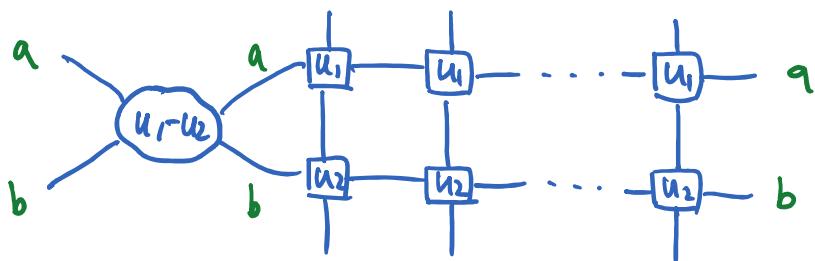
key: off-diagonal term in the monodromy

$$T_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a$$

$$B(u) = \left\langle \uparrow \right| L_{N,a}(u) \cdots L_{1,a}(u) \left| \downarrow \right\rangle_a$$

Fundamental commutation relations (FCR) (also called RTT relations)

$$R_{ab}(u_1 - u_2) T_a(u_1) T_b(u_2) = T_b(u_2) T_a(u_1) R_{ab}(u_1 - u_2)$$



Proof : use Yang-Baxter equation !

Matrix form of FCR / RTT :

$$T_a(u_1) = \begin{pmatrix} A(u_1) & B(u_1) \\ C(u_1) & D(u_1) \end{pmatrix}_a \otimes \mathbb{1}_b = \begin{pmatrix} A(u_1) & 0 & B(u_1) & 0 \\ 0 & A(u_1) & 0 & B(u_1) \\ C(u_1) & 0 & D(u_1) & 0 \\ 0 & C(u_1) & 0 & D(u_1) \end{pmatrix}_{a,b}$$

$$T_b(u_2) = \mathbb{1}_a \otimes \begin{pmatrix} A(u_2) & B(u_2) \\ C(u_2) & D(u_2) \end{pmatrix}_b = \begin{pmatrix} A(u_2) & B(u_2) & 0 & 0 \\ C(u_2) & D(u_2) & 0 & 0 \\ 0 & 0 & A(u_2) & B(u_2) \\ 0 & 0 & C(u_2) & D(u_2) \end{pmatrix}_{a,b}$$

$$R_{ab}(u_1, u_2) = (u_1 - u_2) \mathbb{1}_a \otimes \mathbb{1}_b + i P_{a,b}$$

$$= \begin{pmatrix} u_1 - u_2 + i & 0 & 0 & 0 \\ 0 & u_1 - u_2 & i & 0 \\ 0 & i & u_1 - u_2 & 0 \\ 0 & 0 & 0 & u_1 - u_2 + i \end{pmatrix}_{a,b}$$

FCR/RTT leads to a few operator identities:

$$B(u_1) B(u_2) = B(u_2) B(u_1) \quad \leftarrow \text{commuting!}$$

$$A(u_1) B(u_2) = \frac{u_1 - u_2 - i}{u_1 - u_2} B(u_2) A(u_1) + \frac{i}{u_1 - u_2} B(u_1) A(u_2)$$

$$D(u_1) B(u_2) = \frac{u_1 - u_2 + i}{u_1 - u_2} B(u_2) D(u_1) - \frac{i}{u_1 - u_2} B(u_1) D(u_2)$$

exchange spectral parameters!

$$\underbrace{|u_1, \dots, u_m\rangle}_{\text{Bethe vector}} = B(u_1) B(u_2) \cdots B(u_m) |0\rangle$$

Bethe vector

We require that $|u_1, \dots, u_m\rangle$ is an eigenvector

of $t(u) = A(u) + D(u)$.

$$A(u) |u_1, \dots, u_M\rangle$$

$$= \underbrace{A(u)}_{\text{use FCR}} B(u_1) \cdots B(u_M) |0\rangle$$

$$= \frac{u-u_1-i}{u-u_1} B(u_1) \underbrace{A(u) B(u_2) \cdots B(u_M)}_{\text{use FCR}} |0\rangle$$

$$+ \frac{i}{u-u_1} B(u) \underbrace{A(u_1) B(u_2) \cdots B(u_M)}_{\text{use FCR}} |0\rangle$$

[note that in the last step
A(u) exchanged spectral
parameter with B(u_1)!]

$$= \prod_{n=1}^M \frac{u-u_{n+i}}{u-u_n} B(u_1) B(u_2) \cdots B(u_M) \underbrace{A(u)}_{(u+\frac{i}{\sum})^N |0\rangle} |0\rangle$$

\nearrow
A(u) did NOT exchange
spectral parameter with any B

$$+ \sum_{n=1}^M \underbrace{W_n^A(u, u_1, \dots, u_M)}_{\text{}} B(u_1) \cdots B(u_{n-1}) \underbrace{B(u) B(u_{n+1}) \cdots B(u_M)}_{\text{}} |0\rangle$$

\nearrow
 $\frac{i(u_n + \frac{i}{\sum})^N}{u-u_n} \prod_{j=1}^{n-1} \frac{u_n - u_j - i}{u_n - u_j}$

n: the place where A(u) and B(u_n)
exchange spectral parameters,
and afterwards A(u_n) keeps u_n
as its spectral parameter.

(You may convince
yourself by explicitly

calculating W_1^A and then obtaining W_n^A by substituting $u_1 \rightarrow u_n$.)

Similarly,

$$D(u) |u_1, \dots, u_M\rangle = \left(u - \frac{i}{2}\right)^N \prod_{n=1}^M \frac{u - u_n + i}{u - u_n} |u_1, \dots, u_M\rangle$$

$$\begin{aligned} &+ \sum_{n=1}^M \underbrace{W_n^D(u, u_1, \dots, u_M)}_{\text{!!}} |u_1, \dots, u_{n-1}, u, u_{n+1}, \dots, u_M\rangle \\ &\quad - \frac{-i(u_n - \frac{i}{2})^N}{u - u_n} \prod_{j=1 (\neq n)}^M \frac{u_n - u_j + i}{u_n - u_j} \end{aligned}$$

$$t(u) |u_1, \dots, u_M\rangle$$

$$= [A(u) + D(u)] |u_1, \dots, u_M\rangle$$

$$= \left[\left(u + \frac{i}{2}\right)^N \prod_{n=1}^M \frac{u - u_n - i}{u - u_n} + \left(u - \frac{i}{2}\right)^N \prod_{n=1}^M \frac{u - u_n + \frac{i}{2}}{u - u_n} \right] |u_1, \dots, u_M\rangle$$

$$+ \sum_{n=1}^M \left[W_n^A(u, u_1, \dots, u_M) + W_n^D(u, u_1, \dots, u_M) \right] \underbrace{|u_1, \dots, u_{n-1}, u, u_{n+1}, \dots, u_M\rangle}_{\text{"unwanted term"}}$$

So we require that

$$W_n^A(u, u_1, \dots, u_M) + W_n^D(u, u_1, \dots, u_M) = 0$$

$$\Rightarrow \left(\frac{u_n + \frac{i}{2}}{u_n - \frac{i}{2}} \right)^N = \prod_{j=1 (\neq n)}^M \frac{u_n - u_j + i}{u_n - u_j - i}$$

a set of equations determining u_j ($j=1, \dots, M$)

Bethe ansatz equations (BAE) !

Solutions of BAE provide an eigenvector of $t(u)$:

$$t(u) |u_1, \dots, u_M\rangle = \left[(u + \frac{i}{2})^N \prod_{n=1}^M \frac{u - u_n - i}{u - u_n} + (u - \frac{i}{2})^N \prod_{n=1}^M \frac{u - u_n + i}{u - u_n} \right] |u_1, \dots, u_M\rangle$$

u_1, \dots, u_M satisfy Bethe ansatz equations.

For finite-size chain, BAE usually need to be solved numerically.

Momentum:

$$\begin{aligned} t(u = \frac{i}{2}) &= i^N P_{12} P_{23} \cdots P_{N-1, N} \\ &= i^N \hat{T} \\ &= i^N e^{i\hat{P}} \quad \text{momentum operator} \end{aligned}$$

$$t(u = \frac{i}{2}) |u_1, \dots, u_M\rangle = i^N \prod_{n=1}^M \frac{u_n + \frac{i}{2}}{u_n - \frac{i}{2}} |u_1, \dots, u_M\rangle$$

$$\begin{aligned} \hat{P} |u_1, \dots, u_M\rangle &= -i \ln \left(\prod_{n=1}^M \frac{u_n + \frac{i}{2}}{u_n - \frac{i}{2}} \right) |u_1, \dots, u_M\rangle \\ &= \sum_{n=1}^M i \ln \left(\frac{u_n - \frac{i}{2}}{u_n + \frac{i}{2}} \right) |u_1, \dots, u_M\rangle \end{aligned}$$

$$\begin{aligned} P &= \sum_{n=1}^M P_n \quad \text{with} \quad P_n = i \ln \frac{u_n - \frac{i}{2}}{u_n + \frac{i}{2}} \Rightarrow u_n = \frac{1}{2} \cot \frac{P_n}{2} \\ &\quad \text{[momentum carried by individual "magnon" created by } B(u_n)] \end{aligned}$$

Energy :

$$\underbrace{\left[\tilde{i} \frac{d}{du} \ln t(u) \right]}_{\text{}} \Big|_{u=\frac{i}{2}} |u_1, \dots, u_M\rangle$$

$$\sum_{j=1}^N (2 \vec{s}_j \cdot \vec{s}_{j+1} + \frac{1}{2})$$

see page ⑧

$$= \tilde{i} \frac{d}{du} \ln \left[\left(u + \frac{i}{2} \right)^N \prod_{n=1}^M \frac{u - u_n - i}{u - u_n} + \left(u - \frac{i}{2} \right)^N \prod_{n=1}^M \frac{u - u_n + \frac{i}{2}}{u - u_n} \right] \Big|_{u=\frac{i}{2}} |u_1, \dots, u_M\rangle$$

$$= \tilde{i} \frac{N \cdot i^{N-1} \prod_{n=1}^M \frac{u_n + \frac{i}{2}}{u_n - \frac{i}{2}} + \tilde{i}^N \sum_{n=1}^M \frac{i}{(u - u_n)^2} \prod_{j=1 (j \neq n)}^M \frac{u - u_j - i}{u - u_j}}{\tilde{i}^N \prod_{n=1}^M \frac{u_n + \frac{i}{2}}{u_n - \frac{i}{2}}} \Big|_{u=\frac{i}{2}} |u_1, \dots, u_M\rangle$$

$$= \left[N - \sum_{n=1}^M \frac{1}{(u_n - \frac{i}{2})^2} \cdot \frac{u_n - \frac{i}{2}}{u_n + \frac{i}{2}} \right] \Big|_{u_1, \dots, u_M} |u_1, \dots, u_M\rangle$$

$$= \left(N - \sum_{n=1}^M \frac{1}{u_n^2 + 1/4} \right) \Big|_{u_1, \dots, u_M} |u_1, \dots, u_M\rangle$$

$$\underbrace{\sum_{n=1}^M}_{\text{}} \varepsilon_n = - \frac{1}{u_n^2 + 1/4}$$

$E = \sum_{n=1}^M \varepsilon_n \Rightarrow$ sum over single-magnon energies !

Together with $u_n = \frac{1}{2} \cot \frac{p_n}{2}$, we obtain the magnon dispersion:

$$\varepsilon_n = -4 \sin^2 \frac{p_n}{2} = 2 (\cos p_n - 1)$$

↪ gapless for $p_n \rightarrow 0$!

- Ground state appears for $M=N/2$ (no magnetization).

The elementary excitation for the spin- $1/2$ AFM chain is spinon, not magnon. But the above calculation at least shows that there are gapless excitations, supporting Haldane's conjecture.

Some subtleties about Bethe Ansatz:

- 1) Do Bethe vectors form a complete Hilbert Space?
 (together with degenerate states created by $\sum_{j=1}^N \hat{S}_j^\pm$)

Believed to be true, but rigorously proved only for several models.

- 2) How to calculate static/dynamical correlation functions?

Static: $\langle \vec{S}_j \cdot \vec{S}_{j+l} \rangle$, only achieved for small l
 (asymptotics not available)

→ need other methods (e.g. bosonization)

dynamical: successful for spin- $1/2$ AFM with very good agreement with experiments!

(thanks to good understandings of overlaps between Bethe vectors and the behavior of Bethe strings $\{u_j\}$)

However, difficult to generalize to other models solved by Bethe Ansatz.