

§2. Symmetry breaking and phase transitions

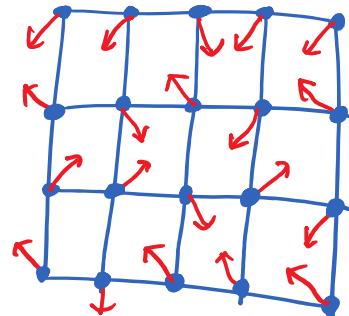
* Spontaneous symmetry breaking

- Classical XY model on a d-dimensional cubic Lattice

$$H = -J \sum_{\langle i,j \rangle} \vec{I}_i \cdot \vec{I}_j$$

$\hookrightarrow i \text{ & } j \text{ are nearest neighbors}$

$$\vec{I}_i = (I_i^x, I_i^y) = I(\cos \theta_i, \sin \theta_i)$$



$$\theta_j \in [0, 2\pi] \text{ and } \theta_j \text{ continuous}$$

(restricting to

$\theta = 0, \pi \Rightarrow$ Ising model

$\theta = 0, \pm \frac{2\pi}{3} \Rightarrow$ Z₃ clock

: model)

$$\begin{aligned} \vec{I}_i \cdot \vec{I}_j &= I^2 (\cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j) \\ &= I^2 \cos(\theta_i - \theta_j) \end{aligned}$$

$$\underline{\text{U(1) symmetry}} : \theta_j \rightarrow \theta_j + \alpha \Rightarrow H \text{ unchanged}$$

Consider $J > 0$ case

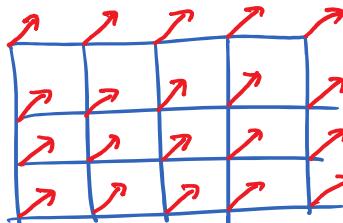
(Note that it's equivalent to $J < 0$ case for bipartite lattices,

by defining $\theta_j \in B$ sublattice $\rightarrow \theta_j \in B$ sublattice $+ \pi$ with

$\theta_j \in A$ sublattice unchanged .)

Lowest-energy configurations ($J > 0$):

$\cos(\theta_i - \theta_j) = +1 \Rightarrow$ same θ_i on every site
infinitely many choices due to $U(1)$ symmetry



ferromagnetic (FM) order

Q: Does this pattern (FM order) survive at $T > 0$?

Need to quantify the FM order ...

Magnetization:

$$\vec{M}_i = \langle \vec{I}_i \rangle = \frac{1}{Z} \int \prod_{j=1}^N d\theta_j \vec{I}_i e^{-\beta H(\{\theta\})}$$

But M_i would be zero due to the $U(1)$ symmetry:

$$\begin{aligned} \langle \vec{I}_i \rangle &= \left(I \underbrace{\langle \cos \theta_i \rangle}_{\parallel}, I \underbrace{\langle \sin \theta_i \rangle}_{\langle \sin(\theta_i + \pi) \rangle} \right) \\ &\quad (\text{replace } \theta_i \text{ by } \theta_i + \pi, \\ &\quad \quad \quad H \text{ unchanged}) \\ &\quad \parallel \\ &\quad - \langle \cos \theta_i \rangle \\ &= - (I \langle \cos \theta_i \rangle, I \langle \sin \theta_i \rangle) \\ &= - \langle \vec{I}_i \rangle \end{aligned}$$

$$\Rightarrow \langle \vec{I}_i \rangle = 0$$

Not the correct way to quantify FM order ...

Correlation function:

$$C(\vec{R}_i, \vec{R}_j) = C(\vec{R}_i - \vec{R}_j) \equiv \langle \vec{I}_i \cdot \vec{I}_j \rangle$$

↑
translation symmetry

$$= I^2 \langle \cos(\theta_i - \theta_j) \rangle$$

\vec{R}_j : coordinate vector of site j

$$T=0 : \quad \theta_i \text{ all equal} \Rightarrow C(\vec{R}_i - \vec{R}_j) = I^2 \quad \forall i, j$$

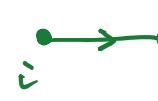
long-range correlation!

- High temperature expansion

$$T \text{ large} \Rightarrow \beta = 1/T \text{ small}$$

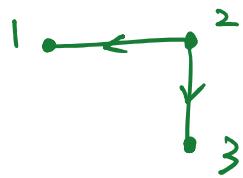
$$\begin{aligned} e^{-\beta H} &= e^{\beta J I^2 \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)} \\ &= 1 + \beta J I^2 \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) + \dots \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} (\beta J I^2)^m \left[\sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \right]^m \\ &\quad \underbrace{\qquad\qquad\qquad}_{\substack{\parallel \\ \sum_{\langle i,j \rangle} \frac{e^{i(\theta_i - \theta_j)} + e^{-i(\theta_j - \theta_i)}}{2}}} \end{aligned}$$

Graphical representation:

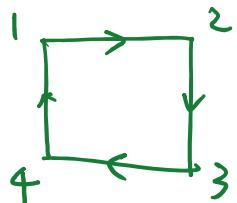


$$= e^{i(\theta_i - \theta_j)}$$

Represent the expansion terms as graphs :



$$e^{i(\theta_2 - \theta_1)} e^{i(\theta_2 - \theta_3)}$$



$$e^{i(\theta_1 - \theta_2)} e^{i(\theta_2 - \theta_3)} e^{i(\theta_3 - \theta_4)} e^{i(\theta_4 - \theta_1)} = 1$$

Note that $\int_0^{2\pi} d\theta e^{im\theta} = \begin{cases} 0, & m = \pm 1, \pm 2, \dots \\ 2\pi, & m = 0 \end{cases}$

$$\begin{aligned} C(\vec{R}_i - \vec{R}_j) &= \frac{1}{N} \int \prod_{k=1}^N d\theta_k I^2 \cos(\theta_i - \theta_j) e^{-\beta H} \\ &= \frac{\int \prod_{k=1}^N d\theta_k \frac{I^2}{2} [e^{i(\theta_i - \theta_j)} + e^{i(\theta_j - \theta_i)}] \sum_{m=0}^{\infty} \frac{1}{m!} \left[\frac{\beta J I^2}{2} \sum_{i,j} (e^{i(\theta_i - \theta_j)} + e^{i(\theta_j - \theta_i)}) \right]^m}{\int \prod_{k=1}^N d\theta_k \sum_{m=0}^{\infty} \frac{1}{m!} \left[\frac{\beta J I^2}{2} \sum_{i,j} (e^{i(\theta_i - \theta_j)} + e^{i(\theta_j - \theta_i)}) \right]^m} \\ &= \frac{\int d\theta \left[\text{Diagram } i \xrightarrow{\text{open}} \xrightarrow{\text{open}} j + \text{Diagram } i \xrightarrow{\text{open}} \xrightarrow{\text{open}} j \text{ with a self-loop on } j \right. + \dots \left. \right]}{\int d\theta \left[1 + \text{Diagram } \square \text{ with one arrow} + \text{Diagram } \square \text{ with two arrows} + \dots \right]} \end{aligned}$$

For the denominator, only oriented closed loops contribute.

For the numerator, only oriented open strings between i and j contribute (oriented close loops may appear together).

In the numerator, **shortest** open strings are favored, while longer ones are suppressed by the factor $\beta J I^2 \ll 1$.

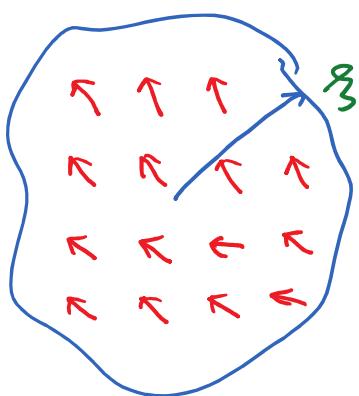
Length of the **shortest** open string $\sim |\vec{R}_i - \vec{R}_j|$

$$\begin{aligned} \Rightarrow C(\vec{R}_i - \vec{R}_j) &\sim \left(\frac{\beta J I^2}{2} \right)^{|\vec{R}_i - \vec{R}_j|} \\ &= e^{-\ln\left(\frac{2}{\beta J I^2}\right)|\vec{R}_i - \vec{R}_j|} \\ &= e^{-|\vec{R}_i - \vec{R}_j|/\xi} \end{aligned}$$

Exponentially decay with correlation length

$$\xi = \frac{1}{\ln\left(\frac{2}{\beta J I^2}\right)} = \frac{1}{\ln\left(\frac{2T}{J I^2}\right)}$$

increases when T decreases



With a region of radius ξ , most spins point to the same direction.

Phase diagram : $T=0$: FM long-range order
 high T : short-range order with exponentially decaying correlation.

phase transition at $T = T_c$ (divergence of ξ) ?

Difficult to answer without numerics for $d \geq 2$

$d=1$: exact solution via transfer matrix method (exercise)

$d=1$: Exponentially decaying correlation at $T > 0$, no long-range order

$d=2$: $T < T_c$, powerlaw decaying correlation,

$$C(\vec{R}_i - \vec{R}_j) \sim \frac{1}{|\vec{R}_i - \vec{R}_j|^\eta}$$

quasi-long-range order (algebraic order)

$T = T_c$: Kosterlitz-Thouless (KT) transition

$d \geq 3$: $T < T_c$, long-range order

$$C(\vec{R}_i - \vec{R}_j) \sim \text{const.} \text{ for } |\vec{R}_i - \vec{R}_j| \rightarrow \infty$$

$T = T_c$: Landau's theory of phase transition
 + Wilson's renormalization group

- Order parameter

$d \gg 3$: $T < T_c$, almost all spins point to the same direction (rigidity)

Q: How to correctly calculate the magnetization?

A: Add a small magnetic field ...

$$H = -J I^2 \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) + \sum_i \vec{h} \cdot \vec{I}_i$$

$$\vec{h} = (\underline{h_x}, 0), h_x > 0$$

x-direction is now "preferred".

Magnetization:

$$\langle \vec{M}_i \rangle = \underbrace{\lim_{h_x \rightarrow 0}}_{\text{order of two limits}} \underbrace{\lim_{N \rightarrow \infty}}_{\text{cannot be changed!}} \langle \vec{I}_i \rangle \neq 0 \quad \text{for FM long-range ordered phase}$$

order of two limits

cannot be changed!

(otherwise $\langle \vec{M}_i \rangle$ vanishes)

This procedure is known as the Bogoliubov quasi-average.

* Goldstone mode

- Quantum rotor model

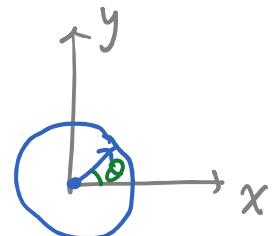
$$H = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

make θ_j operators, but still have no dynamics,
need a conjugate operator for $\hat{\theta}_j$...

Plane rotor in quantum mechanics:

$$\hat{H} = \frac{\hat{L}_z^2}{2C}$$

→ "moment of inertia"



$$\hat{L}_z = -i \frac{\partial}{\partial \theta} \Rightarrow \hat{H} = -\frac{1}{2C} \partial_\theta^2$$

Eigenfunctions $\psi_m = \frac{1}{\sqrt{2\pi}} e^{im\theta}, \quad m=0, \pm 1, \pm 2, \dots$

Eigenenergies $E_m = \frac{m^2}{2C}.$

$[\hat{\theta}, \hat{L}_z] = i \quad (\text{choose } \hbar=1)$

(similar to

\hat{x} and \hat{p})

But be careful!

$\hat{\theta}|\theta\rangle = \theta|\theta\rangle, \quad \theta \in [0, 2\pi)$

$\hat{L}_z|m\rangle = m|m\rangle, \quad m=0, \pm 1, \pm 2, \dots$

$\psi_m(\theta) = \langle \theta | m \rangle = \frac{1}{\sqrt{2\pi}} e^{im\theta}$

You will see
there is an issue
in the path integral...

Quantum rotor model :

$$\hat{H} = \sum_j \frac{\hat{L}_j^2}{2c} - J \sum_{\langle i,j \rangle} \cos(\hat{\theta}_i - \hat{\theta}_j)$$

$$[\hat{\theta}_j, \hat{L}_l] = i$$

sometimes called
"Josephson coupling"

Similar to the single-particle quantum mechanics (\hat{x} & \hat{p}):

$$\begin{aligned} Z &= \text{Tr } e^{-\beta \hat{H}} \\ &= \int D\theta DL e^{-\int_0^\beta d\tau \left[-i \sum_{j=1}^N L_j \dot{\theta}_j + H(\{L, \theta\}) \right]} \end{aligned}$$

Gaussian integration over L :

$$\begin{aligned} &\int DL e^{-\int_0^\beta d\tau \left[\sum_j \frac{1}{2c} L_j^2 - i \sum_j L_j \dot{\theta}_j \right]} \\ &= \int DL e^{-\int_0^\beta d\tau \left[\sum_j \frac{1}{2c} (L_j - i c \dot{\theta}_j)^2 + \sum_j \frac{c}{2} \dot{\theta}_j^2 \right]} \\ &= \text{const.} \times e^{-\int_0^\beta d\tau \sum_j \frac{c}{2} \dot{\theta}_j^2} \\ \Rightarrow Z &\propto \int D\theta e^{-\int_0^\beta d\tau \left[\sum_j \frac{c}{2} \dot{\theta}_j^2 - J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \right]} \end{aligned}$$

Caution: The result is correct. However, note that

$\theta \in [0, 2\pi]$, so the particle is confined on a circle
and thus eigenvalues of \hat{L} are discrete (unlike \hat{p})
So functional integration over L is not justified...

More careful derivation with small time-slices :

(consider a single rotor for simplifying notation)

$$\hat{H} = \frac{\hat{L}^2}{2C} + f(\hat{\theta})$$

$$Z = \text{Tr } e^{-\beta \hat{H}}$$

$$= \int \prod_{k=0}^{N-1} d\theta_k \langle \theta_0 | e^{-\Delta\tau \hat{H}} | \theta_{N-1} \rangle \dots \langle \theta_1 | e^{-\Delta\tau \hat{H}} | \theta_0 \rangle,$$

$$\langle \theta_{k+1} | e^{-\Delta\tau \hat{H}} | \theta_k \rangle = \sum_{m_k=-\infty}^{\infty} \underbrace{\langle \theta_{k+1} | m_k \rangle}_{\text{discrete sum, not integral}} \underbrace{\langle m_k | e^{-\Delta\tau \hat{H}} | \theta_k \rangle}_{\text{in the case with } \hat{x} \text{ and } \hat{p}}$$

$$\stackrel{\text{||}}{\quad} \frac{1}{\sqrt{2\pi}} e^{im_k \theta_{k+1}} \stackrel{\text{||}}{\quad} \langle m_k | (1 - \Delta\tau \hat{H}) | \theta_k \rangle$$

$$\stackrel{\text{||}}{\quad} [1 - \Delta\tau H(m_k, \theta_k)] \langle m_k | \theta_k \rangle$$

$$\stackrel{\text{||}}{\quad} e^{-\Delta\tau H(m_k, \theta_k)} \frac{1}{\sqrt{2\pi}} e^{-im_k \theta_k}$$

$$= \sum_{m_k=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{im_k (\theta_{k+1} - \theta_k) - \frac{\Delta\tau}{2C} m_k^2 - \Delta\tau f(\theta_k)}$$

$$= \int_{-\infty}^{\infty} dx \sum_{m_k=-\infty}^{\infty} \underbrace{\delta(x - m_k)}_{\text{||}} \frac{1}{\sqrt{2\pi}} e^{-ix(\theta_{k+1} - \theta_k) - \frac{\Delta\tau}{2C} x^2 - \Delta\tau f(\theta_k)}$$

$$\stackrel{\text{||}}{\quad} \sum_{n=-\infty}^{\infty} e^{2\pi i n x} \quad (\text{Poisson summation formula})$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} dx}_{\text{Gaussian integral!}} e^{-\frac{\Delta\tau}{2C} [x + \frac{iC}{\Delta\tau} (\theta_{k+1} - \theta_k - 2\pi n)]^2} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{C^2}{2\Delta\tau} (\theta_{k+1} - \theta_k - 2\pi n)^2 - \Delta\tau f(\theta_k)}$$

$$= \sqrt{\frac{c}{2\pi\Delta\tau}} \sum_{n=-\infty}^{\infty} e^{-\frac{c^2}{2\Delta\tau}(\theta_{k+1}-\theta_k - 2\pi n)^2 - \Delta\tau f(\theta_k)}$$

\downarrow
 $\dot{\theta}(\tau) \approx \Delta\tau$ for $N \rightarrow \infty$,
 only $n=0$ term contributes
 in the limit $\Delta\tau \rightarrow 0$
 (other terms $\sim e^{-1/\Delta\tau} \rightarrow 0$)

$$\simeq \sqrt{\frac{c}{2\pi\Delta\tau}} e^{-\frac{c^2}{2\Delta\tau}(\theta_{k+1}-\theta_k)^2 - \Delta\tau f(\theta_k)}$$

Collect the small time-slices :

$$Z = \int D\theta e^{-\int_0^\beta d\tau \left[\frac{c}{2} \dot{\theta}(\tau)^2 + f(\theta) \right]},$$

which agrees with the naive Gaussian integration over L
 in page ⑨.