

§2. Symmetry breaking and phase transitions

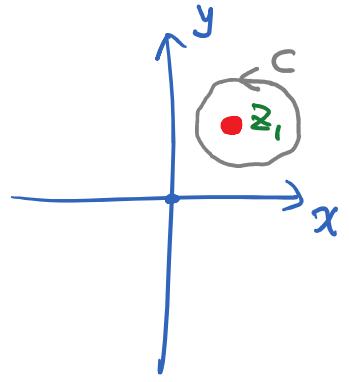
* Kosterlitz - Thouless transition (cont'd)

- Semiclassical approach from the continuum limit

$$H = \frac{J}{2} \int d^2 \vec{R} [\vec{\nabla} \theta(\vec{R})]^2$$

single vortex: $\theta(\vec{R}) = m \operatorname{Im} \ln(z - z_1)$

winding number (charge) position of the vortex core



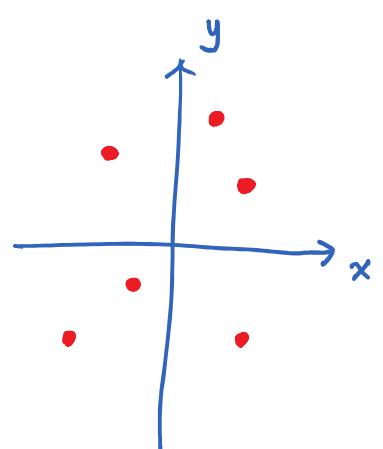
$$\oint_C \vec{\nabla} \theta \cdot d\vec{l} = 2\pi m \quad m \in \mathbb{Z}$$

$$E_{\text{vortex}} = m^2 \pi J \ln \frac{R_c}{a}$$

Many vortices: labeled by (m_r, z_r)

charge position

$$\begin{aligned} \theta_{\text{vortex}}(\vec{R}) &= \operatorname{Im} \ln \left[\prod_r (\vec{z} - \vec{z}_r)^{m_r} \right] \\ &= \sum_r m_r \operatorname{Im} \ln(\vec{z} - \vec{z}_r) \end{aligned}$$



assumption: vortex cores do not overlap.

$$\begin{aligned}
 E_{\text{vortex}} &= \frac{J}{2} \int d^2 \vec{R} \left[\vec{\nabla} \theta_{\text{vortex}}(\vec{R}) \right]^2 \\
 &= \frac{J}{2} \int d^2 \vec{R} \left[\vec{\nabla} \sum_r m_r \text{Im} \ln(z - z_r) \right]^2 \\
 &= \frac{J}{2} \int d^2 \vec{R} \left[\sum_r m_r \vec{\nabla} \text{Im} \ln(z - z_r) \right]^2 \\
 &= \frac{J}{2} \int d^2 \vec{R} \sum_{r,r'} m_r m_{r'} \left[\vec{\nabla} \text{Im} \ln(z - z_r) \cdot \vec{\nabla} \text{Im} \ln(z - z_{r'}) \right] \\
 &\quad \xrightarrow{r=r' \text{ and } r \neq r'} \\
 &= \frac{J}{2} \sum_r m_r^2 \underbrace{\int d^2 \vec{R} \left[\vec{\nabla} \text{Im} \ln(z - z_r) \right]^2}_{\parallel 2\pi \ln \frac{R_c}{a} \text{ (c.f. single-vortex energy)}} \\
 &\quad + \frac{J}{2} \sum_{r \neq r'} m_r m_{r'} \underbrace{\int d^2 \vec{R} \vec{\nabla} \text{Im} \ln(z - z_r) \cdot \vec{\nabla} \text{Im} \ln(z - z_{r'})}_{\parallel \text{exercise}} \\
 &\quad \text{Inter-vortex interactions : } V_{r,r'} = -4\pi \ln \frac{|z_r - z_{r'}|}{a} \\
 &= \pi J \ln \left(\frac{R_c}{a} \right) \sum_r m_r^2 - 2\pi J \sum_{r \neq r'} m_r \ln \frac{|z_r - z_{r'}|}{a} m_{r'} \\
 &\quad \xrightarrow{d=2 \text{ Coulomb potential}} \\
 \Rightarrow E_{\text{vortex}} &= \pi J \ln \left(\frac{R_c}{a} \right) \sum_r m_r^2 - 2\pi J \sum_{r \neq r'} m_r \ln \frac{|\vec{r} - \vec{r}'|}{a} m_{r'}
 \end{aligned}$$

Spin-wave fluctuations on top of many vortices:

$$\Theta(\vec{R}) = \Theta_{\text{vortex}}(\vec{R}) + \underbrace{\delta\Theta(\vec{R})}_{||}$$

$\Theta_{\text{SW}}(\vec{R})$: no singularities
(spin waves)

$$H = \int d^2\vec{R} \left(\vec{\nabla}\Theta_{\text{vortex}} + \vec{\nabla}\Theta_{\text{sw}} \right)^2$$

$$= \int d^2\vec{R} \left[\vec{\nabla}\Theta_{\text{vortex}}(\vec{R}) \right]^2 + \left[\vec{\nabla}\Theta_{\text{sw}}(\vec{R}) \right]^2$$

$$+ \underbrace{\int d^2\vec{R} 2 \vec{\nabla}\Theta_{\text{vortex}}(\vec{R}) \cdot \vec{\nabla}\Theta_{\text{sw}}(\vec{R})}_{|| \text{ exercise} \quad 0}$$

$$= H_{\text{vortex}} + H_{\text{sw}}$$

$$\Theta_{\text{sw}}(\vec{R}) = \int \frac{d^2\vec{k}}{(2\pi)^2} \Theta_{\text{sw}}(\vec{k}) e^{i\vec{k} \cdot \vec{R}}$$

Rebuild partition function:

$$Z = Z_{\text{sw}} \sum_{n=0}^{\infty} \frac{1}{n!} e^{-\beta \pi J \ln(\frac{R_c}{a}) \sum_r m_r^2} \\ \times e^{2\pi \beta J \sum_{r+r'} m_r \ln \frac{|r-r'|}{a} m_{r'}}$$

of vortices

$d=2$ Coulomb gas: This agrees with the derivation based on the Villain's model and justifies the identification of m 's as vortices.

- Effective theory for spin waves

Go back to the lattice derivation based on the Villain model:

$$\begin{aligned}
 Z_V &= \int \prod_r d\phi_r \sum_{\{m_r\}} e^{-\frac{1}{2\beta J} \sum_{r,r'} (\phi_r - \phi_{r'})^2 + 2\pi i \sum_r m_r \phi_r} \\
 &= \underbrace{\int \prod_r d\phi_r \sum_{\{m_r\}} e^{-\frac{1}{2\beta J} \sum_{r,r'} \phi_r (G^{-1})_{rr'} \phi_{r'}}}_{\text{integration over } \phi} + 2\pi i \sum_r m_r \phi_r \\
 &= Z_{SW} \sum_{\{m_r\}=-\infty}^{\infty} e^{-2\pi^2 \beta J \sum_{r,r'} m_r G_{r,r'} m_{r'}} \quad \text{part } Z_{SW}
 \end{aligned}$$

$G_{r,r'} \approx -\frac{1}{2\pi} \ln \frac{|\vec{r} - \vec{r}'|}{a} - \frac{1}{4} + G(0)$

 $\underbrace{G'_{r,r'}}_{\ln \frac{R_c}{a}}$

 (divergent part)

$$\begin{aligned}
 \Rightarrow Z &= Z_{SW} \sum_{\{m_r\}=-\infty}^{\infty} e^{-2\pi^2 \beta J G(0) \left(\sum_r m_r \right)^2} \quad \text{favors } \sum_r m_r = 0 \\
 &\times e^{-2\pi^2 \beta J \sum_{r,r'} m_r G'_{r,r'} m_{r'}} \quad \text{at low T} \\
 &\quad \downarrow \\
 &\quad r=r', G'_{r=r'}=0 \text{ because by definition} \\
 &\quad \text{the nonvanishing part of } G_{r=r'} \text{ is} \\
 &\quad \text{included in } G(0)
 \end{aligned}$$

$$\begin{aligned}
 \sum_{r,r'} m_r G'_{r,r'} m_{r'} &= \sum_{r \neq r'} m_r G'_{r,r'} m_{r'} \\
 &= -\frac{1}{4} \underbrace{\sum_{r \neq r'} m_r m_{r'}}_{\|} - \frac{1}{2\pi} \sum_{r \neq r'} m_r \ln \frac{|\vec{r} - \vec{r}'|}{a} m_{r'} \\
 &\quad \| \\
 &\quad \sum_r m_r \left(\sum_{r'} m_{r'} - m_r \right) \\
 &\quad \| \\
 &\quad \left(\sum_r m_r \right)^2 - \sum_r m_r^2 \\
 &\quad \downarrow \text{use charge neutral condition} \\
 &\quad - \sum_r m_r^2
 \end{aligned}$$

After using $\sum_r m_r = 0$ (valid at low T) :

$$\begin{aligned}
 Z &\simeq Z_{SW} e^{-2\pi^2 \beta J \sum_{r,r'} m_r G'_{r,r'} m_{r'}} \\
 &\simeq Z_{SW} e^{-\frac{\pi^2}{2} \beta J \sum_r m_r^2 + \pi \beta J \sum_{r \neq r'} m_r \ln \frac{|\vec{r} - \vec{r}'|}{a} m_{r'}} \\
 &\quad \downarrow \\
 &\quad \ln y = -\frac{\pi^2}{2} \beta J \quad (y: \text{fugacity}) \\
 &\quad \Rightarrow y = e^{-\frac{\pi^2}{2} \beta J} \\
 &\quad y \text{ small at low } T, \quad y > 0 \\
 &\quad \Rightarrow m_r = 0 \text{ favored} \\
 &\quad \text{dominant fluctuations: } m_r = \pm 1
 \end{aligned}$$

It's still difficult to calculate the above statistical sum.
 But the insight that $m_r = 0, \pm 1$ are dominant is still very useful.

To use this information, we go one step backwards, where the ϕ field has not been integrated out.

spin waves

This allows us to study how vortices change the effective theory of spin waves, especially when $T \simeq T_c$, by including the most important vortex configurations: $m_r = 0, \pm 1$

$$Z \simeq \prod_r d\phi_r \sum_{\{m_r\} = 0, \pm 1} e^{-\frac{1}{2\beta J} \sum_{r,r'} (\phi_r - \phi_{r'})^2 + 2\pi i \sum_r m_r \phi_r + \ln y \sum_r m_r^2}$$

↑
add the

fugacity by hand

↓
integration over
 ϕ_r renormalizes y

(We cannot reliably predict
the fugacity from the original
lattice model)

$$\sum_{\{m_r\} = 0, \pm 1} e^{2\pi i \sum_r m_r \phi_r + \ln y \sum_r m_r^2}$$

$$= \sum_{\{m_r\} = 0, \pm 1} \prod_r e^{2\pi i m_r \phi_r + \ln y \cdot m_r^2}$$

$$= \prod_r \sum_{m_r = 0, \pm 1} e^{2\pi i m_r \phi_r + \ln y \cdot m_r^2}$$

$$= \prod_r (1 + e^{2\pi i \phi_r \cdot y} + e^{-2\pi i \phi_r \cdot y})$$

$$= \prod_r [1 + y \cos(2\pi \phi_r)] \quad \text{y small at low T}$$

$$\simeq \prod_r e^{y \cos(2\pi \phi_r)} = e^{y \sum_r \cos(2\pi \phi_r)}$$

$$\Rightarrow Z = \int \prod_r d\phi_r e^{-\frac{1}{2\beta J} \sum_{r,r'} (\phi_r - \phi_{r'})^2 + 2y \sum_r \cos(2\pi\phi_r)}$$

Spin waves now experience a periodic potential caused by the vortices!

$$S[\phi] = \frac{1}{2\beta J} \sum_{r,r'} (\phi_r - \phi_{r'})^2 - 2y \sum_r \cos(2\pi\phi_r)$$

continuum limit: $a \rightarrow 0$

$$\begin{aligned} \sum_{r,r'} (\phi_r - \phi_{r'})^2 &\rightarrow a^2 \sum_r [(\partial_x \phi_r)^2 + (\partial_y \phi_r)^2] \\ &\rightarrow \int d^2 \vec{r} [\vec{\nabla} \phi(\vec{r})]^2 \end{aligned}$$

$$\sum_r \cos(2\pi\phi_r) \rightarrow \frac{1}{a^2} \int d^2 \vec{r} \cos[2\pi\phi(\vec{r})]$$

$$\Rightarrow Z = \int D\phi(\vec{r}) e^{-\int d^2 \vec{r} \left[\frac{1}{2\beta J} (\vec{\nabla} \phi)^2 - \frac{2y}{a^2} \cos(2\pi\phi(\vec{r})) \right]}$$

rescale. $\phi' = \frac{\phi}{\sqrt{\beta J}}$

$$= \int D\phi'(\vec{r}) e^{-\int d^2 \vec{r} \left[\frac{1}{2} (\vec{\nabla} \phi')^2 - 2y' \cos(2\pi \sqrt{\beta J} \phi'(\vec{r})) \right]}$$

\Downarrow
 $S_{\text{sine-Gordon}}[\phi']$

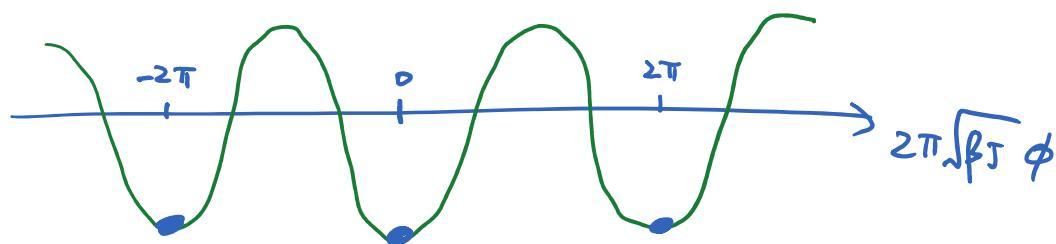
This is the famous Sine-Gordon model!

Effective theory :

$$Z = \int D\phi(\vec{r}) e^{-\int d^2\vec{r} \left[\frac{1}{2}(\nabla\phi)^2 - y \cos(2\pi\sqrt{\beta J}\phi(\vec{r})) \right]}$$

critical point:
 $2\pi\sqrt{\beta_c J} = \sqrt{8\pi}$
 $\Rightarrow T_c = \frac{1}{\beta_c} = \frac{\pi}{2} J$

Role of vortex : periodic potential for spin waves



Q: Is the ϕ -field pinned at the minima?

A: It depends on how large $2\pi\sqrt{\beta J}$ is.

critical point $\beta_c = \frac{2}{\pi J}$ ($T_c = \frac{\pi}{2} J$)

1) $T < T_c$: ϕ is wave-like

(ignore the cos-potential)

2) $T > T_c$: ϕ is pinned in the minima

Renormalization group analysis needed to show this.