

## §2. Symmetry breaking and phase transitions

\* Kosterlitz - Thouless transition (cont'd)

Remarks:

1) Many-vortex configuration  $(m_r, z_r)$ 

$$\rho e^{i\theta_{\text{vortex}}} = \prod_r (z - z_r)^{m_r}$$

$$\begin{aligned} \Rightarrow \theta_{\text{vortex}} &= \text{Im} \ln \left[ \prod_r (z - z_r)^{m_r} \right] \\ &= \text{Im} \sum_r \ln (z - z_r)^{m_r} \\ &= \text{Im} \sum_r m_r \ln (z - z_r) \\ &= \sum_r m_r \text{Im} \ln (z - z_r) \end{aligned}$$

2)  $d=2$  Coulomb gas

$$\begin{aligned} Z &= Z_{\text{sw}} \sum_{\{m_r\}=-\infty}^{\infty} e^{-2\pi^2 \beta J G(0) \left( \sum_r m_r \right)^2} \\ &\times e^{-\frac{\pi^2}{2} \beta J \sum_r m_r^2 + \pi \beta J \sum_{r,r'} m_r \ln \frac{|\vec{r} - \vec{r}'|}{a} m_{r'}} \end{aligned}$$

Low  $T$  (large  $\beta$ ):  $\sum_r m_r = 0$ , charge neutrality  
 $G(0) \sim \ln\left(\frac{R_c}{a}\right)$

$m_r$  small, dominant configurations

$$m_r = 0, \pm 1$$

potential  $V = -\pi\beta J \sum_{r \neq r'} m_r \ln \frac{|\vec{r} - \vec{r}'|}{a} m_{r'}$

Consider  $m_1 > 0, m_2 > 0$

$V_{12} = 0$  if  $|\vec{r}_1 - \vec{r}_2| = a$  (Can be viewed as an overall shift)

$V_{12} < 0$  if  $|\vec{r}_1 - \vec{r}_2| > a$  (Push  $m_2$  away, electric field produced by  $m_1$  has done some work on  $m_2$ , because of the repulsion.)

This justifies the sign.

Alternatively, calculate via

$$\vec{E}_1 = \frac{m_1}{2\pi r \cdot \epsilon_0} \hat{e}_r$$

( $m_1$  at origin)

— Sine-Gordon model

$$Z \simeq \int \prod_r d\phi_r e^{-\frac{1}{2\beta J} \sum_{\langle r, r' \rangle} (\phi_r - \phi_{r'})^2}$$

$$\times \sum_{\{m_r\}} e^{\ln y \cdot \sum_r m_r^2 + 2\pi i \sum_r m_r \phi_r}$$

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$$\parallel \sum_{\{m_r\}} \prod_r e^{\ln y \cdot m_r^2 + 2\pi i m_r \phi_r}$$

$$\parallel \prod_r \sum_{m_r=0, \pm 1, \dots} e^{\ln y \cdot m_r^2 + 2\pi i m_r \phi_r}$$

$$\parallel \prod_r [1 + 2y \cos(2\pi \phi_r) + 2y^4 \cos(4\pi \phi_r) + \dots]$$

$\parallel$  fugacity  $0 < y \ll 1$

$$e^{2y \sum_r \cos(2\pi \phi_r)}$$

Vortices with  $m_r = \pm 2, \dots$  produce  $\cos(4\pi\phi_r), \dots$ .  
 But they are suppressed by the small fugacity  $y \ll 1$ .  
 Furthermore, we will see they are less important than  $\cos(2\pi\phi_r)$  from  $m_r = \pm 1$  (in the renormalization group sense).

$$Z \simeq \int \prod_r \pi d\phi_r e^{-\frac{1}{2\beta J} \sum_{\langle r, r' \rangle} (\phi_r - \phi_{r'})^2 + 2y \sum_r \cos(2\pi\phi_r)}$$

$$\xrightarrow{a \rightarrow 0} \int \mathcal{D}\phi(\vec{r}) e^{-\int d^2\vec{r} \left[ \frac{1}{2\beta J} (\vec{\nabla}\phi)^2 - \frac{2y}{a^2} \cos(2\pi\phi) \right]}$$

rescale:  $\phi' = \frac{1}{\sqrt{\beta J}} \phi$

$$= \int \mathcal{D}\phi'(\vec{r}) e^{-\int d^2\vec{r} \left[ \frac{1}{2} (\vec{\nabla}\phi')^2 - \frac{2y}{a^2} \cos(2\pi\sqrt{\beta J}\phi') \right]}$$

sine-Gordon model

critical point:  $2\pi\sqrt{\beta J} = \sqrt{8\pi} \Rightarrow \beta_c = \frac{2}{\pi J} \quad (T_c = \frac{\pi}{2} J)$

- 1)  $T < T_c$ , cos-term can be neglected, irrelevant in the renormalization group (RG) sense
- 2)  $T > T_c$ , cos-term drastically changes the physics (no matter how small  $y$  is), relevant perturbation
- 3)  $T = T_c$ , critical point, KT transition, cos-term marginal

— Renormalization group (RG)

$$Z = \int D\phi(\vec{r}) e^{-S[\phi]}$$

$$S = S_0 + g \int d^d \vec{r} A(\vec{r})$$

critical theory  
(scale invariant,  
powerlaw decaying  
correlations, e.g.  
massless bosons)

perturbation

spatial dimensions

(space-time dimension  
for quantum problems  
with dynamic critical exponent

$z=1$ )

$$\langle A(\vec{r}) A(\vec{r}') \rangle_0 \sim \frac{1}{|\vec{r} - \vec{r}'|^{2h}}$$

$h$ : scaling dimension of  $A(\vec{r})$

Theorem:

The perturbation is

}	relevant	if $h < d$
	irrelevant	if $h > d$
	marginal	if $h = d$

(Further subtlety exists, e.g.,  $A(\vec{r})$  must have zero conformal spin in case of conformal field theories, but this is typically fulfilled.)

Example:  $d=2$  sine-Gordon model

$$\begin{aligned} A(\vec{r}) &= \cos(2\pi\sqrt{\beta J} \phi'(\vec{r})) \\ &= \frac{1}{2} \left( e^{i2\pi\sqrt{\beta J} \phi'(\vec{r})} + e^{-i2\pi\sqrt{\beta J} \phi'(\vec{r})} \right) \end{aligned}$$

Scaling dimension of  $A(\vec{r})$  ?

$$\begin{aligned} \langle A(\vec{r}) A(\vec{r}') \rangle_0 &\sim \frac{1}{Z_0} \int \mathcal{D}\phi(\vec{r}) e^{i2\pi\sqrt{\beta J} \phi'(\vec{r})} e^{-i2\pi\sqrt{\beta J} \phi'(\vec{r}')} \\ &\quad \times e^{-\int d^2\vec{r} \frac{1}{2} (\vec{\nabla}\phi'(\vec{r}))^2} \end{aligned}$$

You may use Gaussian integral.

For  $S_0 = \int d^2\vec{r} \frac{1}{2} (\vec{\nabla}\phi')^2$ , one has

$$\langle e^{i\alpha\phi'(\vec{r})} e^{-i\alpha\phi'(\vec{r}')} \rangle \sim \left( \frac{a}{|\vec{r} - \vec{r}'|} \right)^{\frac{\alpha^2}{2\pi}} \rightarrow 2h_\alpha$$

$$h_\alpha = \frac{\alpha^2}{4\pi}$$

$$\begin{aligned} \Rightarrow \langle A(\vec{r}) A(\vec{r}') \rangle &\sim \left( \frac{a}{|\vec{r} - \vec{r}'|} \right)^{2 \frac{(2\pi\sqrt{\beta J})^2}{4\pi}} \\ &= \left( \frac{a}{|\vec{r} - \vec{r}'|} \right)^{2\pi\beta J} \end{aligned}$$

The scaling dimension of  $A(\vec{r})$ :  $h_A = \pi\beta J$

relevant if  $h_A = \pi\beta J < d = 2$

$$\Rightarrow \pi\beta_c J = 2, \quad T_c = \frac{1}{\beta_c} = \frac{\pi}{2} J$$

Consequences for the XY (Villain) model:

1)  $T < T_c$

$$S = \int d^2\vec{r} \frac{1}{2} (\nabla\phi')^2 - g \int d^2\vec{r} \cos(2\pi\sqrt{\beta J}\phi')$$

$$\approx \int d^2\vec{r} \frac{1}{2} (\nabla\phi')^2$$

Earlier spin-wave analysis  
is qualitatively correct.

$$\langle e^{i\theta_j} e^{-i\theta_l} \rangle \sim \frac{1}{|\vec{R}_j - \vec{R}_l|^\eta}$$

Quasi-LRO

(with vanishing  
order parameter)

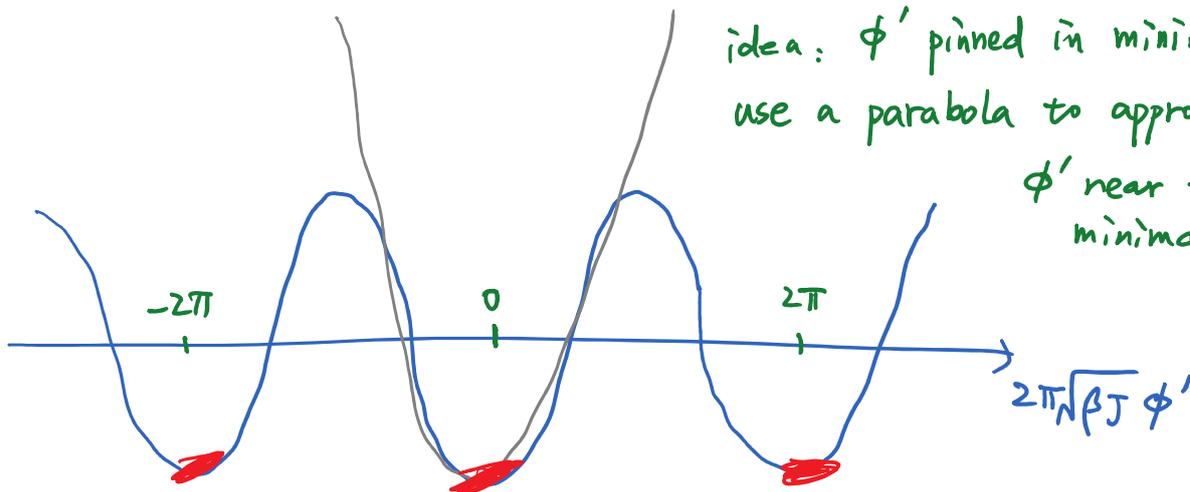
$$\eta = \eta(\beta, J)$$



field theory cannot provide  
reliable estimation of  $\eta$ ,  
except for the KT  
transition point at  $T = T_c$

2)  $T > T_c$

$$S = \int d^2\vec{r} \left[ \frac{1}{2} (\nabla\phi')^2 - 2y' \cos(2\pi\sqrt{\beta J}\phi') \right]$$



idea:  $\phi'$  pinned in minima,  
use a parabola to approximate  
 $\phi'$  near the  
minima.

$$-2y' \omega_3 (2\pi\sqrt{\beta J} \phi') \simeq \frac{1}{2} m^2 (\phi' - \phi_0)^2 + \dots$$

↑  
one of the minima,  
e.g.  $\phi_0 = 0$

$$\Rightarrow S[\phi'] \simeq \int d^2\vec{r} \left[ \frac{1}{2} (\nabla\phi')^2 + \frac{1}{2} m^2 (\phi' - \phi_0)^2 \right]$$

$$= \sum_{\vec{k}} \frac{1}{2} (\vec{k}^2 + m^2) \phi_{-\vec{k}} \phi_{\vec{k}}$$

↑  
mass term

=  
 $\phi_{\vec{k}}^*$

$$\langle e^{i\phi(\vec{r})} e^{-i\phi(\vec{r}')} \rangle \sim e^{-|\vec{r}-\vec{r}'|/\xi}$$

Correlation length  $\xi \sim \frac{1}{m}$

Close to the KT transition point (from  $T > T_c$ ):

Free energy  $f \sim e^{-\frac{\text{const}}{\sqrt{T-T_c}}}$

All derivatives are continuous  $\Rightarrow$  infinite-order transition

No singularities as those in first-order  
or second order phase transitions

Locating the precise KT transition point is very difficult in numerics (No local order parameter, free energy / ground state energy has no singularity, difficult to distinguish powerlaw / exponentially decaying correlations when  $\xi$  is large ...)

$$3) \quad T = T_c$$

universal critical exponent

$$\langle e^{i\theta_j} e^{-i\theta_l} \rangle \sim \text{const.} \left( \frac{a}{|\vec{R}_j - \vec{R}_l|} \right)^{1/4} \underbrace{\left( \ln \frac{|\vec{R}_j - \vec{R}_l|}{a} \right)^{1/8}}$$

extra piece due to the marginal term

(needs a lot of extra work to derive)