

## §1. Many-particle quantum mechanics

## \* Path integral for fermions

— Grassmann coherent state

boson  $|z\rangle = e^{\hat{a}^{\dagger} z} |0\rangle$   
 ↓  
 complex number

fermion  $|\eta\rangle = e^{\hat{f}^{\dagger} \eta} |0\rangle$   
 ↓  
 Grassmann number

$$\langle \bar{\eta} | = \langle 0 | e^{\bar{\eta} \hat{f}}$$

independent from  $\eta$  (there is no need to define "complex conjugate" of Grassmann numbers)

Grassmann numbers:  $\eta, \bar{\eta}$

$$\eta^2 = \bar{\eta}^2 = 0, \quad \eta \bar{\eta} = -\bar{\eta} \eta$$

$$\eta \hat{f} = -\hat{f} \eta, \quad \eta \hat{f}^{\dagger} = -\hat{f}^{\dagger} \eta \quad (\text{similar for } \bar{\eta})$$

(square to zero, mutually anticommute,

and anticommute with fermionic creation/annihilation operators)



Matrix element for Hamiltonian, e.g.  $\hat{H} = \hat{f}^\dagger \hat{f}$

$$\begin{aligned}
 \langle \bar{\eta} | \hat{H} | \eta \rangle &= \langle \bar{\eta} | \hat{f}^\dagger \hat{f} | \eta \rangle \\
 &= \langle \bar{\eta} | \bar{\eta} \eta | \eta \rangle \quad \leftarrow \text{moving } \bar{\eta} \eta \text{ out doesn't} \\
 &= \bar{\eta} \eta \langle \bar{\eta} | \eta \rangle \quad \leftarrow \text{lead to minus sign} \\
 &= \underline{\bar{\eta} \eta} e^{\bar{\eta} \eta} \quad \hat{f}^\dagger \text{ and } \hat{f} \text{ replaced by } \bar{\eta} \text{ and } \eta, \\
 &= \bar{\eta} \eta (1 + \bar{\eta} \eta) \quad \text{with additional factor } e^{\bar{\eta} \eta} \\
 &= \bar{\eta} \eta \quad \text{(generalizable to normal-ordered} \\
 & \quad \quad \quad \text{Hamiltonians in many-fermion cases)}
 \end{aligned}$$

Calculus of Grassmann numbers:

$$\frac{\partial}{\partial \eta} 1 = 0, \quad \frac{\partial}{\partial \eta} \eta = 1 \quad (\text{similar for } \bar{\eta})$$

Example:  $\frac{\partial}{\partial \eta} \eta \bar{\eta} = \bar{\eta}$

$$\frac{\partial}{\partial \bar{\eta}} \eta \bar{\eta} = -\frac{\partial}{\partial \bar{\eta}} \bar{\eta} \eta = -\eta$$

$$\int d\eta = \frac{\partial}{\partial \eta} \quad (\text{similar for } \bar{\eta})$$

Example:  $\int d\eta d\bar{\eta} e^{\bar{\eta} \eta} = \int d\eta d\bar{\eta} (1 + \bar{\eta} \eta)$

$$\begin{aligned}
 &= \frac{\partial}{\partial \eta} \frac{\partial}{\partial \bar{\eta}} \bar{\eta} \eta \\
 &= 1
 \end{aligned}$$

Resolution of identity:

$$\int d\bar{\eta} d\eta e^{-\bar{\eta}\eta} |\eta\rangle\langle\bar{\eta}| = \hat{1}$$

Proof: LHS =  $\int d\bar{\eta} d\eta (1 - \bar{\eta}\eta) (1 + \hat{f}^\dagger\eta) |0\rangle\langle 0| (\bar{\eta}\hat{f} + 1)$

$$= \frac{\partial}{\partial\bar{\eta}} \frac{\partial}{\partial\eta} ( \hat{f}^\dagger\eta |0\rangle\langle 0| \bar{\eta}\hat{f} - \bar{\eta}\eta |0\rangle\langle 0| )$$

$$= \frac{\partial}{\partial\bar{\eta}} \frac{\partial}{\partial\eta} ( \eta\bar{\eta} \underbrace{\hat{f}^\dagger|0\rangle\langle 0|}_{\substack{|| \\ ||\langle 1|}} \hat{f} + \eta\bar{\eta} |0\rangle\langle 0| )$$

$$= |1\rangle\langle 1| + |0\rangle\langle 0|$$

$$= \text{RHS}$$

Trace:  $\text{Tr}(\hat{A}) = \int d\bar{\eta} d\eta \langle -\bar{\eta} | \hat{A} | \eta \rangle e^{-\bar{\eta}\eta}$

$\hat{A}$  involves even number of fermionic operators,  
otherwise  $\text{Tr}(\hat{A})$  is trivially zero (superselection rule)

Proof: RHS =  $\int d\bar{\eta} d\eta \langle 0 | e^{-\bar{\eta}\hat{f}} \hat{A} e^{\hat{f}^\dagger\eta} |0\rangle e^{-\bar{\eta}\eta}$

$$= \int d\bar{\eta} d\eta \langle 0 | (1 + \hat{f}^\dagger\bar{\eta}) \hat{A} (1 + \hat{f}\eta) |0\rangle (1 - \bar{\eta}\eta)$$

$$= \frac{\partial}{\partial\bar{\eta}} \frac{\partial}{\partial\eta} ( \langle 0 | \hat{A} | 0 \rangle \eta\bar{\eta} - \bar{\eta}\eta \langle 0 | \hat{f} \hat{A} \hat{f}^\dagger | 0 \rangle )$$

$$= \langle 0 | \hat{A} | 0 \rangle + \langle 1 | \hat{A} | 1 \rangle$$

$$= \text{LHS}$$

Partition function:



$$Z = \text{Tr} e^{-\beta \hat{H}}$$

$$= \int d\bar{\eta}_0 d\eta_0 \langle -\bar{\eta}_0 | \underbrace{e^{-\Delta\tau \hat{H}} \dots e^{-\Delta\tau \hat{H}}}_{N \text{ time-slices}} | \eta_0 \rangle e^{-\bar{\eta}_0 \eta_0}$$

$N$  time-slices

← inserting resolution of identity

$$= \int \prod_{k=0}^{N-1} d\bar{\eta}_k d\eta_k \langle -\bar{\eta}_0 | e^{-\Delta\tau \hat{H}} | \eta_{N-1} \rangle \langle \bar{\eta}_{N-1} | e^{-\Delta\tau \hat{H}} | \eta_{N-2} \rangle$$

$$\dots \langle \bar{\eta}_1 | e^{-\Delta\tau \hat{H}} | \eta_0 \rangle e^{-\sum_{k=0}^{N-1} \bar{\eta}_k \eta_k}$$

$$\langle \bar{\eta}_{k+1} | e^{-\Delta\tau \hat{H}} | \eta_k \rangle \simeq \langle \bar{\eta}_{k+1} | [1 - \Delta\tau \hat{H}(\hat{f}^\dagger, \hat{f})] | \eta_k \rangle$$

$$= [1 - \Delta\tau H(\bar{\eta}_{k+1}, \eta_k)] \langle \bar{\eta}_{k+1} | \eta_k \rangle$$

$$\simeq e^{-\Delta\tau H(\bar{\eta}_{k+1}, \eta_k)} e^{\bar{\eta}_{k+1} \eta_k}$$

$$= e^{-\Delta\tau H(\bar{\eta}_{k+1}, \eta_k) + \bar{\eta}_{k+1} \eta_k}$$

Be careful with  $\langle -\bar{\eta}_0 | e^{-\Delta\tau \hat{H}} | \eta_{N-1} \rangle$  !

$$\Rightarrow Z = \int \prod_{k=0}^{N-1} d\bar{\eta}_k d\eta_k e^{-\bar{\eta}_0 \eta_{N-1} - \Delta\tau H(-\bar{\eta}_0, \eta_{N-1})}$$

$$\times e^{\sum_{k=0}^{N-2} [\bar{\eta}_{k+1} \eta_k - \Delta\tau H(\bar{\eta}_{k+1}, \eta_k)]} e^{-\sum_{k=0}^{N-1} \bar{\eta}_k \eta_k}$$

Define  $\bar{\eta}_N = -\bar{\eta}_0$

$$\eta_N = -\eta_0$$

antiperiodic boundary condition (antiPBC)

$$= \int \prod_{k=0}^{N-1} d\bar{\eta}_k d\eta_k e^{\sum_{k=0}^{N-1} [-\bar{\eta}_{k+1} (\eta_{k+1} - \eta_k) - \Delta\tau H(\bar{\eta}_{k+1}, \eta_k)]}$$

In the limit  $\Delta\tau \rightarrow 0$   
 $N \rightarrow \infty$  :

$$\lim_{N \rightarrow \infty} \int \prod_{k=0}^{N-1} d\bar{\eta}_k d\eta_k \rightarrow \int \mathcal{D}\bar{\eta}(\tau) \mathcal{D}\eta(\tau)$$

$$\eta_{k+1} - \eta_k \rightarrow \dot{\eta}(\tau) \Delta\tau$$

$$\Rightarrow Z = \int_{\substack{\eta(0) = -\eta(\beta), \\ \bar{\eta}(0) = -\bar{\eta}(\beta)}} \mathcal{D}\bar{\eta}(\tau) \mathcal{D}\eta(\tau) e^{-S[\bar{\eta}, \eta]}$$

$$S[\bar{\eta}, \eta] = \int_0^\beta d\tau [\bar{\eta}(\tau) \partial_\tau \eta(\tau) + H(\bar{\eta}(\tau), \eta(\tau))]$$

Fourier transform in imaginary-time domain:

$$\begin{cases} \eta(\tau) = \frac{1}{\sqrt{\beta}} \sum_{i\omega_n} e^{-i\omega_n \tau} \eta(i\omega_n) \\ \eta(i\omega_n) = \frac{1}{\sqrt{\beta}} \int_0^\beta d\tau e^{i\omega_n \tau} \eta(\tau) \end{cases}$$

$$\eta(\tau=0) = -\eta(\tau=\beta)$$

$$\text{antiperiodicity} \Rightarrow e^{i\omega_n \beta} = -1$$

$$\omega_n = \pm \frac{\pi}{\beta}, \pm \frac{3\pi}{\beta}, \dots \quad \text{fermionic Matsubara frequency!}$$

$$= \frac{2\pi}{\beta} \left( n + \frac{1}{2} \right), \quad n \text{ integer}$$

Generalization to many fermions:

$$\hat{f}_j, \hat{f}_j^+ \rightarrow \eta_j, \bar{\eta}_j$$

$$|\{\eta\}\rangle = e^{\sum_{j=1}^M \hat{f}_j^+ \eta_j} |0\rangle = \prod_{j=1}^M (1 + \hat{f}_j^+ \eta_j) |0\rangle$$

$$\Rightarrow \hat{f}_j |\{\eta\}\rangle = \eta_j |\{\eta\}\rangle$$

$\vdots$  all steps follow previous derivations

$$Z = \int_{\text{antiPBC}} D\bar{\eta}(\tau) D\eta(\tau) e^{-S[\bar{\eta}, \eta]}$$

$$S[\bar{\eta}, \eta] = \int_0^\beta d\tau \left[ \sum_{j=1}^M \bar{\eta}_j(\tau) \partial_\tau \eta_j(\tau) + H(\bar{\eta}, \eta) \right]$$

obtained from normal-ordered  $\hat{H}(\hat{f}^+, \hat{f})$

## \* Quantum spins

— Algebra (angular momentum,  $SU(2)$ )

$$\hat{S}^x, \hat{S}^y, \hat{S}^z$$

$$[\hat{S}^a, \hat{S}^b] = i\epsilon_{abc} \hat{S}^c, \quad a, b, c = x, y, z$$

raising & lowering operators:

$$\hat{S}^{\pm} = \hat{S}^x \pm i\hat{S}^y$$

total-spin operator:

$$\hat{S}^2 = (\hat{S}^x)^2 + (\hat{S}^y)^2 + (\hat{S}^z)^2 = \frac{1}{2} (\hat{S}^+ \hat{S}^- + \hat{S}^- \hat{S}^+) + (\hat{S}^z)^2$$

Hilbert space:  $|S, m\rangle$ ,  $m = S, S-1, \dots, -S$

integer or half-integer  
(0, 1, 2, ...)    ( $\frac{1}{2}, \frac{3}{2}, \dots$ )

$$\begin{cases} \hat{S}^2 |S, m\rangle = S(S+1) |S, m\rangle \\ \hat{S}^z |S, m\rangle = m |S, m\rangle \end{cases}$$

$$\sum_{m=-S}^S |S, m\rangle \langle S, m| = \hat{1}$$

Example: spin-1/2 ( $S = 1/2$ )

$$\begin{cases} |1/2, 1/2\rangle = |\uparrow\rangle \\ |1/2, -1/2\rangle = |\downarrow\rangle \end{cases}$$

$$\hat{S}^a = \frac{1}{2} \hat{\sigma}^a, \quad \begin{cases} \hat{S}^+ = |\uparrow\rangle \langle \downarrow| \\ \hat{S}^- = |\downarrow\rangle \langle \uparrow| \end{cases}$$

Pauli matrices

Many spins :

$$[\hat{S}_j^a, \hat{S}_l^b] = \delta_{j,l} i \epsilon_{abc} \hat{S}_j^c \quad j, l = 1, \dots, N$$

↳ spin operators at different sites commute!  
"bosonic" nature

Hilbert space spanned by

$$|s, m_1\rangle \otimes |s, m_2\rangle \otimes \dots \otimes |s, m_N\rangle$$

↑  
dimension  $(2s+1)^N$  (assume spin- $s$  for each site)

- Schwinger boson representation

$$\hat{a}, \hat{a}^\dagger \quad \& \quad \hat{b}, \hat{b}^\dagger \rightarrow [\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1$$

map spins to bosons!

$$\left\{ \begin{array}{l} \hat{S}^+ = \hat{a}^\dagger \hat{b} \\ \hat{S}^- = \hat{b}^\dagger \hat{a} \\ \hat{S}^z = \frac{1}{2} (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) = \frac{1}{2} (\hat{n}_a - \hat{n}_b) \end{array} \right.$$