

§1. Many-particle quantum mechanics

* Quantum spins

- Schwinger boson representation

$$\hat{a}, \hat{a}^\dagger \text{ & } \hat{b}, \hat{b}^\dagger$$

$$\left\{ \begin{array}{l} \hat{S}^+ = \hat{a}^\dagger \hat{b} \\ \hat{S}^- = \hat{b}^\dagger \hat{a} \\ \hat{S}^z = \frac{1}{2} (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) = \frac{1}{2} (\hat{n}_a - \hat{n}_b) \end{array} \right.$$

$$\Rightarrow [\hat{S}^+, \hat{S}^-] = \hat{a}^\dagger \hat{b} \hat{b}^\dagger \hat{a} - \hat{b}^\dagger \hat{a} \hat{a}^\dagger \hat{b}$$

$$= \hat{n}_a (1 + \hat{b}^\dagger \hat{b}) - \hat{n}_b (1 + \hat{a}^\dagger \hat{a})$$

$$= \hat{n}_a - \hat{n}_b$$

$$= 2 \hat{S}^z$$

$$[\hat{S}^\pm, \hat{S}^\pm] = \mp \hat{S}^\pm$$

SU(2) commutation relations
indeed preserved!

$$\begin{aligned} \hat{S}^2 &= \frac{1}{2} (\hat{S}^+ \hat{S}^- + \hat{S}^- \hat{S}^+) + (\hat{S}^z)^2 \\ &= \frac{1}{4} (\hat{n}_a + \hat{n}_b)(\hat{n}_a + \hat{n}_b + 2) \end{aligned}$$

Hilbert space: $|\hat{n}_a, \hat{n}_b\rangle = \frac{1}{\sqrt{\hat{n}_a! \hat{n}_b!}} (\hat{a}^\dagger)^{\hat{n}_a} (\hat{b}^\dagger)^{\hat{n}_b} |0\rangle$

$$\left\{ \begin{array}{l} \hat{S}^2 |\hat{n}_a, \hat{n}_b\rangle = \frac{1}{2} (\hat{n}_a - \hat{n}_b) |\hat{n}_a, \hat{n}_b\rangle \\ \hat{S}^2 |\hat{n}_a, \hat{n}_b\rangle = \frac{1}{4} (\hat{n}_a + \hat{n}_b)(\hat{n}_a + \hat{n}_b + 2) |\hat{n}_a, \hat{n}_b\rangle \end{array} \right.$$

$\hat{a}|0\rangle = \hat{b}|0\rangle = 0$

Compare with

$$\begin{cases} \hat{S}^z |s, m\rangle = m |s, m\rangle \\ \hat{\vec{S}}^2 |s, m\rangle = s(s+1) |s, m\rangle \end{cases}$$

$$\Rightarrow \begin{cases} m = \frac{1}{2}(n_a - n_b) \\ S = \frac{1}{2}(n_a + n_b) \end{cases} \quad \xrightarrow{\hbar=1 \text{ assumed}}$$

$$|s, m\rangle = \frac{(\hat{a}^+)^{n_a} (\hat{b}^+)^{n_b}}{\sqrt{n_a! n_b!}} = \frac{(\hat{a}^+)^{s+m} (\hat{b}^+)^{s-m}}{\sqrt{(s+m)! (s-m)!}} |0\rangle$$

Link between Schwinger boson states and spin states

Constraint: $n_a + n_b = 2S$

Example: spin-1/2 $\Rightarrow |1/2, 1/2\rangle = |\uparrow\rangle = \hat{a}^+ |0\rangle \quad \uparrow$
 $|1/2, -1/2\rangle = |\downarrow\rangle = \hat{b}^+ |0\rangle \quad \downarrow$

spin-1 $\Rightarrow |1, 1\rangle = \frac{1}{\sqrt{2}} (\hat{a}^+)^2 |0\rangle \quad \boxed{\uparrow \uparrow}$

$|1, 0\rangle = \hat{a}^+ \hat{b}^+ |0\rangle \quad \boxed{\uparrow \downarrow}$

$|1, -1\rangle = \frac{1}{\sqrt{2}} (\hat{b}^+)^2 |0\rangle \quad \boxed{\downarrow \downarrow}$

Idea of Schwinger bosons:

Symmetrization of $2S$ spin-1/2's
to form higher spin-S object.

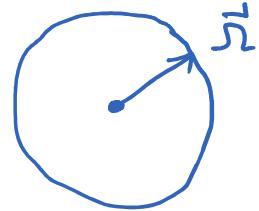
- Spin coherent state

Q: What is the classical picture of a quantum spin?

Classical spin: $|\vec{s}| = 1$

$$\vec{s} = (s^x, s^y, s^z)$$

$$= (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$



A: spin coherent state + path integral

Spin coherent state:

$$|\vec{s}\rangle = \frac{1}{\sqrt{(2S)!}} (z_1 \hat{a}^\dagger + z_2 \hat{b}^\dagger)^{2S} |0\rangle$$

notation will be justified later!

complex numbers

Think of $P_{n_a+n_b=2S}$ $e^{z_1 \hat{a}^\dagger} e^{z_2 \hat{b}^\dagger} |0\rangle$

projector enforcing the constraint $n_a+n_b=2S$

$$= P_{n_a+n_b=2S} e^{z_1 \hat{a}^\dagger + z_2 \hat{b}^\dagger} |0\rangle$$

$$= P_{n_a+n_b=2S} \left[1 + (z_1 \hat{a}^\dagger + z_2 \hat{b}^\dagger) + \frac{1}{2!} (z_1 \hat{a}^\dagger + z_2 \hat{b}^\dagger)^2 + \dots \right] |0\rangle$$

pick up the $2S$ -th order in the Taylor expansion

$$= \frac{1}{(2S)!} (z_1 \hat{a}^\dagger + z_2 \hat{b}^\dagger)^{2S} |0\rangle$$

$$\begin{aligned}
 |\vec{\Omega}\rangle &= \frac{1}{\sqrt{(2s)!}} (z_1 \hat{a}^+ + z_2 \hat{b}^+)^{2s} |0\rangle \\
 &= \frac{1}{\sqrt{(2s)!}} \sum_{m=-s}^s \underbrace{\frac{(2s)!}{(s+m)!(s-m)!}}_{\text{binomial coefficient}} z_1^{s+m} z_2^{s-m} (\hat{a}^+)^{s+m} (\hat{b}^+)^{s-m} |0\rangle \\
 &\quad // \\
 &= \sum_{m=-s}^s \underbrace{\sqrt{\frac{(2s)!}{(s+m)!(s-m)!}}}_{\binom{2s}{s+m}} z_1^{s+m} z_2^{s-m} |s, m\rangle \\
 &= \sum_{m=-s}^s \sqrt{\binom{2s}{s+m}} z_1^{s+m} z_2^{s-m} |s, m\rangle
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \langle \vec{\Omega} | \vec{\Omega} \rangle &= \sum_{m=-s}^s \binom{2s}{s+m} (z_1^* z_1)^{s+m} (z_2^* z_2)^{s-m} \\
 &= (|z_1|^2 + |z_2|^2)^{2s}
 \end{aligned}$$

$$\text{Require } \langle \vec{\Omega} | \vec{\Omega} \rangle = 1 \Rightarrow |z_1|^2 + |z_2|^2 = 1$$

Parametrization:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = e^{i\chi} \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}$$

Mathematically,
 $\mathbb{CP}^1 = S^2 / U(1)$

overall phase in $|\vec{\Omega}\rangle$,
gauge degrees of freedom

\Rightarrow set $\chi = \frac{\phi}{2}$,
fix a gauge

Thus,

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

Overcompleteness relation: (Proof: exercise)

$$\frac{2S+1}{4\pi} \int d\vec{s} |\vec{s}\rangle \langle \vec{s}| = \sum_{m=-S}^S |s, m\rangle \langle s, m| = \hat{1}$$

||

$$\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$$

Haar measure of $SU(2)$: $\frac{2S+1}{4\pi} d\vec{s} = \frac{2S+1}{4\pi} \sin\theta d\theta d\phi$

Matrix element: (Proof: exercise)

$$\begin{cases} \langle \vec{s} | \hat{s}^x | \vec{s} \rangle = S \sin\theta \cos\phi = S s \Omega^x \\ \langle \vec{s} | \hat{s}^y | \vec{s} \rangle = S \sin\theta \sin\phi = S s \Omega^y \\ \langle \vec{s} | \hat{s}^z | \vec{s} \rangle = S \cos\theta = S s \Omega^z \end{cases}$$

$$\Rightarrow \langle \vec{s} | \vec{s} \cdot \vec{s} | \vec{s} \rangle = S \quad \text{justifies the notation } |\vec{s}\rangle !$$

$|\vec{s}\rangle$ is a fully polarized state in \vec{s} -direction.

Example: spin-1/2 $\Rightarrow |\vec{s}\rangle = (z_1 \hat{a}^+ + z_2 \hat{b}^+) |0\rangle$
 $= z_1 |\uparrow\rangle + z_2 |\downarrow\rangle$

$$\begin{aligned} \langle \vec{s} | \hat{s}^+ | \vec{s} \rangle &= (\underbrace{\langle \uparrow | z_1^* + \langle \downarrow | z_2^*}_{z_1^* z_2}) \hat{s}^+ (z_1 |\uparrow\rangle + \underbrace{z_2 |\downarrow\rangle}_{\sin\theta e^{i\phi}}) \\ &= z_1^* z_2 \quad (\text{note that } \hat{s}^+ = \hat{a}^+ \hat{b} \rightarrow z_1^* z_2) \\ &= \sin\theta \sin\theta e^{i\phi} = \frac{1}{2} \sin\theta e^{i\phi} \end{aligned}$$

$$\begin{aligned} \langle \vec{s} | \hat{s}^- | \vec{s} \rangle &= z_2^* z_1 \quad (\hat{s}^- = \hat{b}^+ \hat{a} \rightarrow z_2^* z_1) \\ &= \frac{1}{2} \sin\theta e^{-i\phi} \quad \Rightarrow \langle \vec{s} | \hat{s}^x | \vec{s} \rangle = \langle \vec{s} | \frac{1}{2} (\hat{s}^+ + \hat{s}^-) | \vec{s} \rangle \\ &= \frac{1}{2} \sin\theta \cos\phi \end{aligned}$$

— Path integral

"Shortcut" to path integral (based on Schwinger bosons):

$$Z = \text{Tr } e^{-\beta \hat{H}}$$

$$= \int_{\alpha(0)=\alpha(\beta)} D\alpha^*(\tau) D\alpha(\tau) D\beta^*(\tau) D\beta(\tau) \delta(\alpha^*(\tau)\alpha(\tau) + \beta^*(\tau)\beta(\tau) - 2S)$$

$$\times e^{-\int_0^\beta d\tau [\alpha^*(\tau)\partial_\tau \alpha(\tau) + \beta^*(\tau)\partial_\tau \beta(\tau) + H(\alpha^*, \alpha, \beta^*, \beta)]}$$

δ -function constrains "paths" in spin- S Hilbert space.

Rescale $\alpha(\tau) = \sqrt{2S} z_1(\tau)$, similar for $\alpha^*(\tau), \beta^*(\tau)$

$$b(\tau) = \sqrt{2S} z_2(\tau)$$

$$Z = \int_{z(0)=z(\beta)} Dz^* Dz e^{-\int_0^\beta d\tau [\underbrace{z_S(z_1^* \partial_\tau z_1 + z_2^* \partial_\tau z_2)}_{\text{Berry's phase}} + H(z^*, z)]}$$

More careful derivation:

$$Z = \text{Tr } e^{-\beta \hat{H}}$$

$$= \text{Tr} (e^{-\Delta\tau \hat{H}} \cdots e^{-\Delta\tau \hat{H}})$$

$$= \left(\frac{2S+1}{4\pi}\right)^N \int \prod_{k=0}^{N-1} d\vec{\eta}_k \langle \vec{\eta}_0 | e^{-\Delta\tau \hat{H}} | \vec{\eta}_{N-1} \rangle \cdots \langle \vec{\eta}_1 | e^{-\Delta\tau \hat{H}} | \vec{\eta}_0 \rangle$$

$$N \rightarrow \infty (\Delta\tau \rightarrow 0) : Z = \int_{\vec{\eta}(0)=\vec{\eta}(\beta)} D\vec{\eta}(\tau) e^{-\int_0^\beta d\tau [\underbrace{\cdots}_{\text{Berry's phase}} + H(\vec{\eta})]}$$

The explicit form of the Berry's phase can be derived from a small time-slice:

$$\langle \vec{s}_{k+1} | e^{-i\tau \hat{H}} | \vec{s}_k \rangle = \dots \quad (\text{exercise})$$

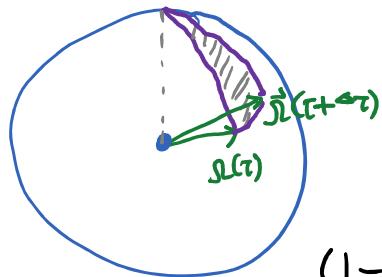
Instead we can also use the "shortcut" result:

$$\begin{aligned} a^* \partial_\tau a + b^* \partial_\tau b &= 2S(z_1^* \partial_\tau z_1 + z_2^* \partial_\tau z_2) \\ &= 2S \left[\underbrace{\cos \frac{\theta}{2} \partial_\tau (\cos \frac{\theta}{2})}_{-\frac{1}{2} \sin^2 \frac{\theta}{2} \partial_\tau \theta} + \underbrace{\sin \frac{\theta}{2} e^{-i\phi} \partial_\tau (\sin \frac{\theta}{2} e^{i\phi})}_{\frac{1}{2} \cos \frac{\theta}{2} (\partial_\tau \theta) e^{i\phi}} \right. \\ &\quad \left. + \sin \frac{\theta}{2} e^{i\phi} i \partial_\tau \phi \right] \\ &= 2S \sin^2 \frac{\theta}{2} i \partial_\tau \phi \\ &= iS(1 - \cos \theta) \partial_\tau \phi \end{aligned}$$

$$\Rightarrow z = \int_{\vec{s}(\alpha) = \vec{s}(\beta)} D\vec{s}(\tau) e^{-S[\vec{s}]}$$

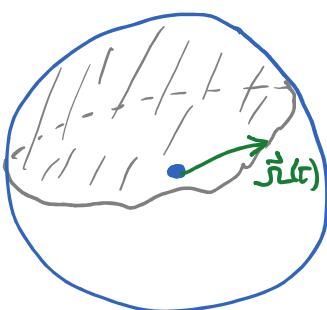
$$S[\vec{s}] = \underbrace{iS \int_0^\beta d\tau (1 - \cos \theta) \dot{\phi}}_{\substack{\text{purely imaginary} \\ \text{Berry's phase}}} + \underbrace{\int_0^\beta d\tau H(\vec{s})}_{\substack{\text{obtained from } \hat{H} \\ \text{by } \vec{s} \rightarrow S\vec{s}}}$$

Geometric picture of the Berry's phase:



Berry's phase \Rightarrow solid angle
on the unit sphere

$$(1 - \cos \theta) \dot{\varphi} \Delta t$$



trajectory $t \in [0, \beta]$

Solid angle (dashed area) determined moduli 4π
(area of the unit sphere)

$$e^{iS \cdot 4\pi} = 1$$

such that the path integral is well-defined.

This is consistent with $S = \text{integer or half-integer!}$

- Many spins

$$\hat{H} = \sum_{j,l} J_{j,l} \hat{\vec{s}}_j \cdot \hat{\vec{s}}_l + \dots$$

$$Z = \int_{\vec{s}(0) = \vec{s}(\beta)} D\vec{s}(\tau) e^{-S[\vec{s}(\tau)]/\hbar} \quad (\text{restore } \hbar \text{ here})$$

$$S[\vec{s}(\tau)] = iS \int_0^\beta d\tau \sum_j (1 - \cos \theta_j) \dot{\varphi}_j + \int_0^\beta H(\vec{s}(\tau)) d\tau$$

$$H(\vec{s}(\tau)) = S^2 \sum_{j,l} J_{j,l} \vec{s}_j \cdot \vec{s}_l$$

Classical limit: Roughly speaking, $S \rightarrow \infty$

$$\Rightarrow \hbar \rightarrow \hbar/S \rightarrow 0$$

$\delta S=0$, "classical" path dominates!

Not entirely true ...

Interference due to the Berry's phase can be very important!

Example: Half-integer spin chains in one dimension
(Haldane's conjecture)

→ will be covered in later lectures ...