

§3. Bosons

* Bose-Einstein Condensation

Remarks:

1) Thermal Green's function:

Imaginary-time evolution of operators and
path integral formulation

See page ③ in boson-1. pdf

2) Analysis of BEC

 $d=3$: see page ⑧ in boson-1. pdf

$$N = \int_0^\infty d\varepsilon \underbrace{\sum_{\mathbf{k}} \delta(\varepsilon - \frac{|\mathbf{k}|^2}{2m})}_{\frac{1}{V} P(\varepsilon)} \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$$

$P(\varepsilon)$: density-of-states

$$\Rightarrow \frac{N}{V} = \int_0^\infty d\varepsilon \frac{P(\varepsilon)}{e^{\beta(\varepsilon - \mu)} - 1}$$

$$d=1, \quad P(\varepsilon) \sim \frac{1}{\sqrt{\varepsilon}}$$

$$d=2, \quad P(\varepsilon) \sim \text{const.}$$

integral diverges for $\mu=0$.
 \Rightarrow (In other words, μ cannot reach 0
 for $T > 0$)

free particle \nearrow
 $\varepsilon_{\mathbf{k}} = \frac{|\mathbf{k}|^2}{2m}$

NO BEC in $d=1$ and $d=2$
 for ideal Bose gas at $T > 0$.

* Bogoliubov's theory of superfluidity

$d=3$, weakly repulsive interaction ($g > 0$)

Lesson from BEC:

$$S_0 = \sum_{\vec{k}, i\omega_n} \left(-i\omega_n + \frac{|\vec{k}|^2}{2m} - \mu \right) \bar{a}(\vec{k}, i\omega_n) a(\vec{k}, i\omega_n)$$

When condensation occurs:

$$a(\vec{k}=0, i\omega_n=0) \equiv a_0 \propto \sqrt{N}$$

$$\bar{a}(\vec{k}=0, i\omega_n=0) \equiv \bar{a}_0 \propto \sqrt{N} \quad (\text{up to a phase})$$

$$N = \underbrace{|a_0|^2}_{\text{condensed part}} + \underbrace{\sum_{\vec{k} \neq 0} n_B\left(\frac{|\vec{k}|^2}{2m}\right)}_{\text{non-condensed part}}$$

This suggests to single out the (potentially) condensed part in $\psi(\vec{r}, \tau)$ and $\bar{\psi}(\vec{r}, \tau)$:

$$\psi(\vec{r}, \tau) = \frac{1}{\sqrt{\beta V}} \sum_{\vec{k}, i\omega_n} a(\vec{k}, i\omega_n) e^{i\vec{k} \cdot \vec{r} - i\omega_n \tau}$$

← $\vec{k}=0, i\omega_n=0$ removed

$$= \frac{1}{\sqrt{\beta V}} a_0 + \frac{1}{\sqrt{\beta V}} \sum'_{\vec{k}, i\omega_n} a(\vec{k}, i\omega_n) e^{i\vec{k} \cdot \vec{r} - i\omega_n \tau}$$

$$= \psi_0 + \underbrace{\psi_1(\vec{r}, \tau)}_{\text{fluctuation field}}$$

$$\bar{\psi}(\vec{r}, \tau) = \bar{\psi}_0 + \bar{\psi}_1(\vec{r}, \tau)$$

Rewrite the action in terms of ψ_0, ψ_1 :

$$\begin{aligned}
 S_0 &= \int_0^\beta d\tau \int d^3\vec{r} \bar{\psi}(\vec{r}, \tau) \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi(\vec{r}, \tau) \\
 &= \int_0^\beta d\tau \int d^3\vec{r} \left[\bar{\psi}_0 + \bar{\psi}_1(\vec{r}, \tau) \right] \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \left[\psi_0 + \psi_1(\vec{r}, \tau) \right] \\
 &= -\beta V \mu |\psi_0|^2 + \int_0^\beta d\tau \int d^3\vec{r} \bar{\psi}_1(\vec{r}, \tau) \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi_1(\vec{r}, \tau) \\
 &\quad + \int_0^\beta d\tau \int d^3\vec{r} \left\{ -\mu \left[\bar{\psi}_0 \psi_1(\vec{r}, \tau) + \psi_0 \bar{\psi}_1(\vec{r}, \tau) \right] \right\} \\
 &\qquad\qquad\qquad \text{terms linear in } \bar{\psi}_1, \psi_1
 \end{aligned}$$

Other terms vanish:

$$\int_0^\beta d\tau \int d^3\vec{r} \bar{\psi}_0 \partial_\tau \psi_1(\vec{r}, \tau) = \int d^3\vec{r} \bar{\psi}_0 \left[\psi_1(\vec{r}, \beta) - \psi_1(\vec{r}, 0) \right]$$

because of the periodicity of bosonic field
in imaginary time direction

$$\int_0^\beta d\tau \int d^3\vec{r} \bar{\psi}_0 \frac{\nabla^2}{2m} \psi_1(\vec{r}, \tau) = \int_0^\beta d\tau \frac{\bar{\psi}_0}{2m} \int d^3\vec{r} \nabla \cdot \nabla \psi_1(\vec{r}, \tau)$$

"integration of total derivatives"

Be careful! Sometimes it's dangerous to drop out
"integration of total-derivatives", because they might
be topological terms (like the winding number of vortices).
But this is not expected to be the case here.

Interaction part:

$$S_1 = \int_0^\beta dt \int d^3\vec{r} \frac{1}{2} g \underbrace{[\bar{\psi}(\vec{r}, \tau) \psi(\vec{r}, \tau)]^2}_{\parallel (\bar{\psi}_0 + \bar{\psi}_1)^2 (\psi_0 + \psi_1)^2}$$

$$\parallel (\bar{\psi}_0^2 + \bar{\psi}_1^2 + 2\bar{\psi}_0 \bar{\psi}_1) (\psi_0^2 + \psi_1^2 + 2\psi_0 \psi_1)$$

$$= \beta V \cdot \frac{1}{2} g |\psi_0|^4$$

$$+ \int_0^\beta dt \int d^3\vec{r} \frac{1}{2} g [\bar{\psi}_0^2 (\psi_1^2 + 2\psi_0 \psi_1) + \psi_0^2 (\bar{\psi}_1^2 + 2\bar{\psi}_0 \bar{\psi}_1) + 4|\psi_0|^2 \bar{\psi}_1 \psi_1] + \underbrace{\text{higher-order terms}}_{\downarrow}$$

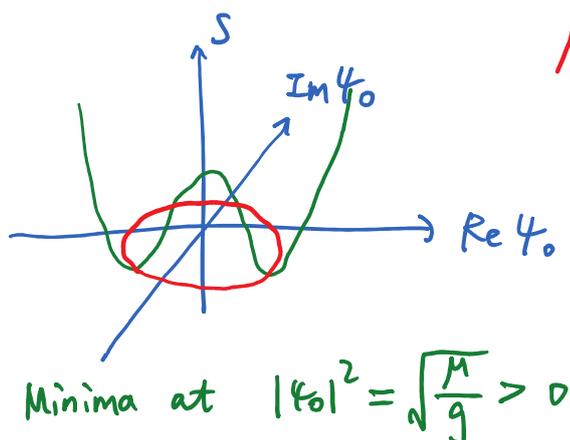
$$\bar{\psi}_0 \bar{\psi}_1 \psi_1^2, \bar{\psi}_1^2 \psi_0, (\bar{\psi}_1 \psi_1)^2$$

(which will be dropped out)
↓
Only Gaussian fluctuations will be considered.

$$\Rightarrow S = S_0 + S_1$$

$$= \beta V (-\mu |\psi_0|^2 + \frac{1}{2} g |\psi_0|^4) + \dots$$

↪ because of repulsive interaction ($g > 0$), $\mu > 0$ can be reached.



Bad drawing ...

You could imagine the bottom of the red-wine bottle.

This is a typical example of symmetry-breaking.

$|\psi_0|^2$ is fixed to $\frac{\mu}{g}$ (mean-field result) due to energetic reason, but the phase of ψ_0 is not fixed.

All phases of ψ_0 have the same energy (due to the fact that the model has an associated $U(1)$ symmetry, which corresponds to particle-number conservation), but the system will just choose one phase (rigidity)

w.l.o.g we choose $\psi_0 = \bar{\psi}_0 = \text{real}$

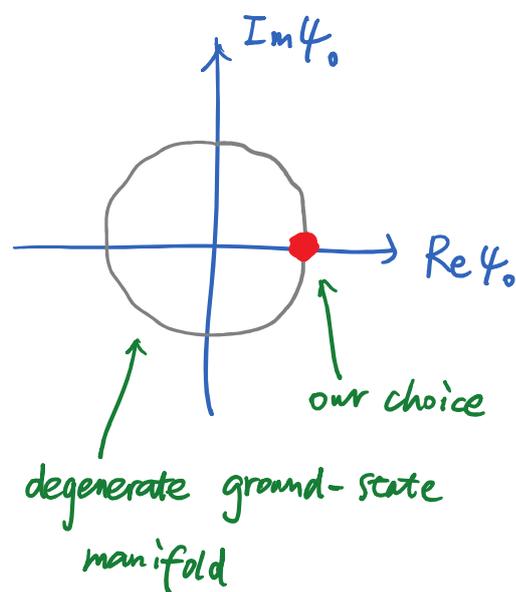
Remarks:

1) Gapless Goldstone boson excitation should appear.

2) Problematic treatment for $d=1$ and $d=2$ at $T>0$

(spontaneous breaking $U(1)$ symmetry not allowed \Leftarrow Mermin-Wagner theorem)
Fluctuations will destroy LRO.

(But this does NOT rule out the possibility of superfluidity for $d < 3$!)



Continue with S by setting $\psi_0 = \bar{\psi}_0 = \sqrt{\frac{M}{g}}$.

Drop the constant term $\beta V(-\mu|\psi_0|^2 + \frac{1}{2}g|\psi_0|^4)$.

Observation: terms linear in $\bar{\psi}_1$ and ψ_1 cancel

$$\int_0^\beta d\tau \int d^3\vec{r} \left[-\mu\psi_0(\bar{\psi}_1 + \psi_1) + \frac{1}{2}g\psi_0^3(2\bar{\psi}_1 + 2\psi_1) \right] = 0$$

from S_0 ,
see ③

$$\frac{M}{g}\psi_0$$

from S_1 ,
see ④

Remaining terms:

$$\begin{aligned} S[\bar{\psi}_1, \psi_1] &= \int_0^\beta d\tau \int d^3\vec{r} \left[\bar{\psi}_1 \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi_1 \right. \\ &\quad \left. + \frac{1}{2}g\psi_0^2 (\bar{\psi}_1^2 + \psi_1^2 + 4\bar{\psi}_1\psi_1) \right] \\ &= \int_0^\beta d\tau \int d^3\vec{r} \left[\bar{\psi}_1 \left(\partial_\tau - \frac{\nabla^2}{2m} \right) \psi_1 + \frac{1}{2}g\psi_0^2 (\bar{\psi}_1 + \psi_1)^2 \right] \end{aligned}$$

The corresponding Hamiltonian would be

$$\hat{H}_{\text{eff}} = \sum_{\vec{k}(\neq 0)} \left[\left(\frac{|\vec{k}|^2}{2m} + g\psi_0^2 \right) \hat{a}_{\vec{k}}^+ \hat{a}_{\vec{k}} + \frac{1}{2}g\psi_0^2 (\hat{a}_{\vec{k}}^+ \hat{a}_{-\vec{k}} + \hat{a}_{-\vec{k}} \hat{a}_{\vec{k}}) \right]$$

$\psi_1, \bar{\psi}_1$ does not
have zero-mode part

Fourier mode of $\hat{\psi}_1^+(\vec{r})$

Diagonalizing \hat{H}_{eff} requires
Bogoliubov transformation.

Below we continue with path integral formulation.

Introduce real components of $\psi, \bar{\psi}$:

$$\begin{cases} \psi_1(\vec{r}, \tau) = A(\vec{r}, \tau) + iP(\vec{r}, \tau) \\ \bar{\psi}_1(\vec{r}, \tau) = A(\vec{r}, \tau) - iP(\vec{r}, \tau) \end{cases}$$

$$S[A, P] = \int_0^\beta d\tau \int d^3\vec{r} \left[(A - iP) \left(\partial_\tau - \frac{\nabla^2}{2m} \right) (A + iP) + \frac{1}{2} g \psi_0^2 (2A)^2 \right]$$

Many terms vanish:

$$\int_0^\beta d\tau A \partial_\tau A = \underbrace{\int_0^\beta d\tau \partial_\tau (A^2)}_{\parallel} - \int_0^\beta d\tau \underbrace{(\partial_\tau A) A}_{\parallel}$$

$$\parallel A^2(\vec{r}, \beta) - A^2(\vec{r}, 0) = 0 \quad \parallel \delta_\tau A$$

$$\Rightarrow \int_0^\beta d\tau A \partial_\tau A = 0$$

$$\text{Similarly, } \int_0^\beta d\tau P \partial_\tau P = 0$$

$$\int d^3\vec{r} (A \nabla^2 P - P \nabla^2 A) = \int d^3\vec{r} \underbrace{\nabla \cdot (A \nabla P - P \nabla A)}_{\parallel} = 0$$

$$\parallel (\nabla A \cdot \nabla P + A \nabla^2 P) - (\nabla P \cdot \nabla A + P \nabla^2 A)$$

$$\parallel A^2 \nabla P - P \nabla^2 A$$

$$S[A, P] = \int_0^\beta d\tau \int d^3\vec{r} \left[\underbrace{iA\partial_\tau P - iP\partial_\tau A}_{\substack{\parallel \\ \partial_\tau(PA) - (\partial_\tau P)A \\ \text{total-derivative, dropped!}}} \right. \\ \left. + A\left(-\frac{\nabla^2}{2m} + 2g\phi_0^2\right)A + P\left(-\frac{\nabla^2}{2m}\right)P \right] \\ = \int_0^\beta d\tau \int d^3\vec{r} \left[\underbrace{2iA\partial_\tau P}_{\substack{\downarrow \\ \text{Berry's phase, indicating } A \text{ and } P \text{ are} \\ \text{conjugate variables}}} + A\left(-\frac{\nabla^2}{2m} + 2g\phi_0^2\right)A + P\left(-\frac{\nabla^2}{2m}\right)P \right]$$

Berry's phase, indicating A and P are conjugate variables

$$= \sum_{\vec{k}, i\omega_n} \left[2iA(-\vec{k}, -i\omega_n) (-i\omega_n) P(\vec{k}, i\omega_n) \right. \\ \left. + A(-\vec{k}, -i\omega_n) \left(\frac{|\vec{k}|^2}{2m} + 2g\phi_0^2 \right) A(\vec{k}, i\omega_n) \right. \\ \left. + P(-\vec{k}, -i\omega_n) \frac{|\vec{k}|^2}{2m} P(\vec{k}, i\omega_n) \right]$$

$$= \sum_{\vec{k}, i\omega_n} \left(A(-\vec{k}, -i\omega_n), P(-\vec{k}, -i\omega_n) \right) \underbrace{\begin{pmatrix} \frac{|\vec{k}|^2}{2m} + 2g\phi_0^2 & \omega_n \\ -\omega_n & \frac{|\vec{k}|^2}{2m} \end{pmatrix}}_{\substack{\parallel \times (-\frac{1}{2}) \\ G^{-1}(\vec{k}, i\omega_n)}}} \begin{pmatrix} A(\vec{k}, i\omega_n) \\ P(\vec{k}, i\omega_n) \end{pmatrix}$$

The pole of $G(\vec{k}, i\omega_n)$ corresponds to excitation spectrum.

In this framework, $G(\vec{k}, i\omega_n)$ corresponds to

$$G(\vec{k}, i\omega_n) = - \left\langle \begin{pmatrix} A(\vec{k}, i\omega_n) \\ P(\vec{k}, i\omega_n) \end{pmatrix} \begin{pmatrix} A(-\vec{k}, -i\omega_n) & P(-\vec{k}, -i\omega_n) \end{pmatrix} \right\rangle$$

$$= - \begin{pmatrix} \langle A(\vec{k}, i\omega_n) A(-\vec{k}, -i\omega_n) \rangle & \langle A(\vec{k}, i\omega_n) P(-\vec{k}, -i\omega_n) \rangle \\ \langle P(\vec{k}, i\omega_n) A(-\vec{k}, -i\omega_n) \rangle & \langle P(\vec{k}, i\omega_n) P(-\vec{k}, -i\omega_n) \rangle \end{pmatrix}$$

You may check this by using Gaussian integrals.

$$G(\vec{k}, i\omega_n) = - \frac{1}{\Sigma} \begin{pmatrix} \frac{|\vec{k}|^2}{2m} + 2g\phi_0^2 & \omega_n \\ -\omega_n & \frac{|\vec{k}|^2}{2m} \end{pmatrix}^{-1}$$

$$= - \frac{1}{\Sigma} \frac{1}{\underbrace{\frac{|\vec{k}|^2}{2m} \left(\frac{|\vec{k}|^2}{2m} + 2g\phi_0^2 \right) + \omega_n^2}} \begin{pmatrix} \frac{|\vec{k}|^2}{2m} & -\omega_n \\ \omega_n & \frac{|\vec{k}|^2}{2m} + 2g\phi_0^2 \end{pmatrix}$$

$$\omega_{\vec{k}}^2 + \omega_n^2 \quad \text{with} \quad \omega_{\vec{k}} = \sqrt{\frac{|\vec{k}|^2}{2m} \left(\frac{|\vec{k}|^2}{2m} + 2g\phi_0^2 \right)}$$

$\omega_{\vec{k}}$ is the excitation spectrum!

You may compare with harmonic oscillator:

$$S = \int_0^{\beta} d\tau \left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2 \right)$$

$$= \sum_{i\omega_n} \frac{1}{2} m (\omega_n^2 + \omega^2) x(-i\omega_n) x(i\omega_n)$$

$$\rightarrow G(i\omega_n) = - \langle x(i\omega_n) x(-i\omega_n) \rangle$$

$$= - \frac{1}{2} \frac{1}{\frac{1}{2} m (\omega_n^2 + \omega^2)}$$

excitation energy

Dispersion relation of excitations:

$$\omega_{\vec{k}} = \sqrt{\frac{|\vec{k}|^2}{2m} \left(\frac{|\vec{k}|^2}{2m} + 2g\varphi_0^2 \right)}$$

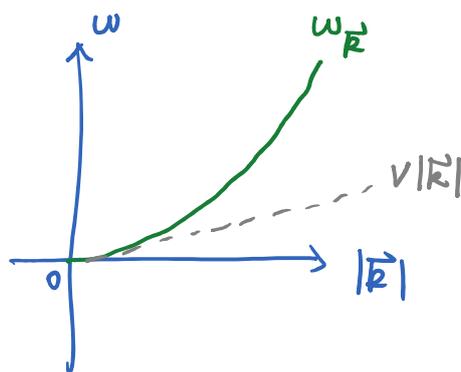
$$= \begin{cases} \varphi_0 \sqrt{\frac{g}{m}} |\vec{k}| & , \text{ small } |\vec{k}| \ll 2\varphi_0 \sqrt{mg} \\ \frac{|\vec{k}|^2}{2m} & , \text{ large } |\vec{k}| \gg 2\varphi_0 \sqrt{mg} \end{cases}$$

long-wavelength limit $|\vec{k}| \rightarrow 0$:

$$\omega_{\vec{k}} \simeq v |\vec{k}|, \quad v = \varphi_0 \sqrt{\frac{g}{m}} \quad (\text{sound velocity})$$

This is indeed the Goldstone mode describing collective excitations, usually called phonon mode.

Remark: Quite different from ideal Bose gas which has ordinary $\omega_{\vec{k}} = \frac{|\vec{k}|^2}{2m}$ bosonic excitations.



Specific heat at low T :

Superfluid $C_v \sim T^3$ (c.f. Debye T^3 law)

ideal Bose gas $C_v \sim T^{3/2}$ (similar to $d=3$ ferromagnets with $|\vec{k}|^2$ -dispersion)