

§ 3. Bosons

*) Bogoliubov's theory of superfluidity (cont'd)

Quick review:

$$\mathcal{Z} = \int D\bar{\psi} D\psi e^{-S}$$

$$S = \int_0^T d\tau \int d^3\vec{r} \left[\bar{\psi}(\vec{r}, \tau) \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi(\vec{r}, \tau) + \frac{1}{2} g (\bar{\psi}(\vec{r}, \tau) \psi(\vec{r}, \tau))^2 \right]$$

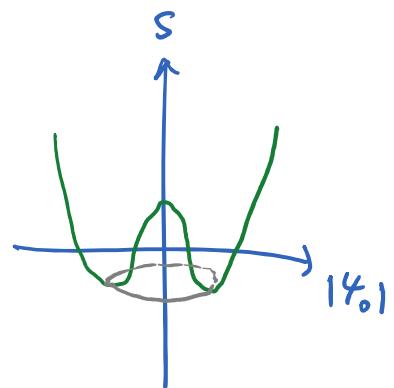
$$\psi(\vec{r}, \tau) = \psi_0 + \psi_1(\vec{r}, \tau)$$

\uparrow \uparrow
 condensed part non-condensed part

$$\Rightarrow S = \beta V \left(-\mu |\psi_0|^2 + \frac{1}{2} g |\psi_0|^4 \right) + \dots$$

$\underbrace{\quad}_{\text{minima}} \quad |\psi_0|^2 = \sqrt{\frac{\mu}{N g}}$

\downarrow
 $U(1)$ symmetry breaking
 \downarrow
 one branch of Goldstone boson



Gaussian fluctuations $(\bar{\psi}_1, \psi_1)$

\Downarrow

$$\omega_{\vec{k}} = \sqrt{\frac{|\vec{k}|^2}{2m} \left(\frac{|\vec{k}|^2}{2m} + 2g|\psi_0|^2 \right)} \xrightarrow{|\vec{k}| \rightarrow 0} V_s |\vec{k}| \quad (\text{phonon})$$

— Validity of Bogoliubov's theory

Q: Under which condition is Bogoliubov's theory
 (mean-field + Gaussian fluctuations)
 a good approximation?

A: Fluctuations are small.

\downarrow
 quantify it!

$$\bar{\psi}(\vec{r}, \tau) = \psi_0 + \psi_1(\vec{r}, \tau) \quad \text{c.f. infinite time-slice picture!}$$

$$\begin{aligned} \text{Particle number: } N &= \int d^3\vec{r} \langle \bar{\psi}(\vec{r}, \tau) \psi(\vec{r}, \tau + \Delta\tau) \rangle \\ &= \sqrt{\psi_0^2} + \int d^3\vec{r} \langle \bar{\psi}_1(\vec{r}, \tau) \psi_1(\vec{r}, \tau + \Delta\tau) \rangle \\ &\stackrel{\text{Gaussian level}}{\simeq} \underbrace{\sqrt{\psi_0^2}}_{N_0} + \underbrace{\int d^3\vec{r} \langle \bar{\psi}_1(\vec{r}, \tau) \psi_1(\vec{r}, \tau + \Delta\tau) \rangle}_{N_1} \\ &\quad \parallel \quad \parallel \\ &\quad (\text{Condensed part}) \quad (\text{fluctuation part}) \end{aligned}$$

$$\begin{aligned} N_1 &= \int d^3\vec{r} \langle \bar{\psi}_1(\vec{r}, \tau) \psi_1(\vec{r}, \tau + \Delta\tau) \rangle_0 \quad \text{from now on, "0" will} \\ &= \frac{1}{\beta} \int_0^\beta d\tau \int d^3\vec{r} \langle \bar{\psi}_1(\vec{r}, \tau) \psi_1(\vec{r}, \tau + \Delta\tau) \rangle \quad \text{be omitted.} \\ &= \frac{1}{\beta} \sum_{\vec{k}, i\omega_n} \langle \bar{\psi}_1(\vec{k}, i\omega_n) \psi_1(\vec{k}, i\omega_n) \rangle e^{-i\omega_n \cdot \Delta\tau} \end{aligned}$$

$$\begin{cases} \psi_1(\vec{r}, \tau) = A(\vec{r}, \tau) + i P(\vec{r}, \tau) \\ \bar{\psi}_1(\vec{r}, \tau) = A(\vec{r}, \tau) - i P(\vec{r}, \tau) \end{cases}$$

$$\Rightarrow \psi_1(\vec{k}, i\omega_n) = \frac{1}{\sqrt{\beta V}} \int_0^\beta d\tau \int d^3 r \underbrace{\psi_1(\vec{r}, \tau)}_{A(\vec{r}, \tau) + i P(\vec{r}, \tau)} e^{i\vec{k} \cdot \vec{r} - i\omega_n \tau} \\ = A(\vec{k}, i\omega_n) + i P(\vec{k}, i\omega_n)$$

$$\bar{\psi}_1(\vec{k}, i\omega_n) = A^*(\vec{k}, i\omega_n) - i P^*(\vec{k}, i\omega_n) = A(-\vec{k}, -i\omega_n) - i P(-\vec{k}, -i\omega_n)$$

Substitute into the expression for N_1 (see last page):

$$N_1 = \frac{1}{\beta} \sum_{\vec{k}, i\omega_n} \left[\langle A(-\vec{k}, -i\omega_n) A(\vec{k}, i\omega_n) \rangle + \langle P(-\vec{k}, -i\omega_n) P(\vec{k}, i\omega_n) \rangle \right. \\ \left. + i \langle A(-\vec{k}, -i\omega_n) P(\vec{k}, -i\omega_n) \rangle - i \langle P(-\vec{k}, -i\omega_n) A(\vec{k}, i\omega_n) \rangle \right] \\ \times e^{-i\omega_n \sigma^-} \quad \begin{matrix} \uparrow \text{see page (9) in boson-2.pdf} \\ \text{for the explicit form of} \\ \text{these four correlators.} \end{matrix}$$

$$= \frac{1}{\beta} \sum_{\vec{k}, i\omega_n} \frac{1}{2} \frac{1}{\omega_n^2 + \omega_{\vec{k}}^2} \left(\frac{|\vec{k}|^2}{2m} + \frac{|\vec{k}|^2}{2m} + 2g q_0^2 + i\omega_n + i\omega_n \right) e^{i\omega_n \sigma^+}$$

$$= \frac{1}{\beta} \sum_{\vec{k}, i\omega_n} \frac{1}{\omega_n^2 + \omega_{\vec{k}}^2} \left(\frac{|\vec{k}|^2}{2m} + g q_0^2 + \underbrace{i\omega_n}_{-\omega_{\vec{k}} + i\omega_n + \omega_{\vec{k}}} \right) e^{i\omega_n \sigma^+}$$

$$= \sum_{\vec{k}} \frac{1}{\beta} \sum_{i\omega_n} \frac{1}{\omega_n^2 + \omega_{\vec{k}}^2} e^{i\omega_n \sigma^+} \underbrace{\left(\frac{|\vec{k}|^2}{2m} + g q_0^2 - \omega_{\vec{k}} \right)}_{\frac{1}{\omega_{\vec{k}}} [n_B(\omega_{\vec{k}}) + \frac{1}{2}]} = \frac{1}{\omega_{\vec{k}}} [n_B(\omega_{\vec{k}}) + \frac{1}{2}]$$

$$+ \sum_{\vec{k}} \frac{-1}{\beta} \sum_{i\omega_n} \frac{1}{i\omega_n - \omega_{\vec{k}}} e^{i\omega_n \sigma^+} \quad \begin{matrix} \leftarrow e^{i\omega_n \sigma^+} \text{ ensures convergence} \\ \text{in the second sum!} \end{matrix}$$

$$N_1 = \sum_{\vec{k}} \left\{ \frac{1}{\omega_{\vec{k}}} \left[n_B(\omega_{\vec{k}}) + \frac{1}{2} \right] \left(\frac{|\vec{k}|^2}{2m} + g 4_0^2 - \omega_{\vec{k}} \right) + n_B(\omega_{\vec{k}}) \right\}$$

Low temperature limit $n_B(\varepsilon) \sim \frac{T}{\varepsilon}$

Consider $T=0$ below :

$$\begin{aligned} N_1 &= \sum_{\vec{k}} \frac{\frac{|\vec{k}|^2}{2m} + g 4_0^2 - \omega_{\vec{k}}}{2\omega_{\vec{k}}} \\ &\rightarrow V \int \frac{d^3 k}{(2\pi)^3} \left(\frac{\frac{|\vec{k}|^2}{2m} + g 4_0^2}{2\omega_{\vec{k}}} - \frac{1}{2} \right) \quad \text{→ } k_0 = 2\sqrt{mg} 4_0 \\ &= \frac{V \cdot 4\pi}{8\pi^3} \frac{1}{2} \int_0^\infty dk k^2 \left(\frac{k^2 + \frac{1}{2}k_0^2}{\sqrt{k^2(k^2+k_0^2)}} - 1 \right) \\ &= \frac{1}{24\pi^2} V k_0^3 \quad \int_0^\infty dx \times \left(\frac{x^2 + \frac{1}{2}}{\sqrt{x^2+1}} - x \right) = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \Rightarrow N &= N_0 + N_1 \\ &= V 4_0^2 + \frac{1}{24\pi^2} V k_0^3 \\ &= V \left[4_0^2 + \underbrace{\frac{1}{3\pi^2} (mg)^{3/2} 4_0^3}_{\text{fluctuations}} \right] \end{aligned}$$

Justification of Bogoliubov's theory:

$$4_0 \sim \sqrt{\frac{N}{V}} \ll (mg)^{-3/2} \quad \text{i) dilute Bose gas}$$

$\frac{N}{V}$: $\overset{\uparrow}{\text{boson density}}$

and/or ii) interaction weak.

*) Lower dimensions : $d=2$ & $d=1$

- $d=2$

Mermin-Wagner theorem : NO symmetry breaking for $T>0$

Develop effective theory for low-energy, long-wavelength degrees of freedom (d.o.f.) !

$$\begin{cases} \hat{q}(\vec{r}, \tau) = \sqrt{\rho(\vec{r}, \tau)} e^{i\theta(\vec{r}, \tau)} \\ \bar{q}(\vec{r}, \tau) = \sqrt{\rho(\vec{r}, \tau)} e^{-i\theta(\vec{r}, \tau)} \end{cases}$$

boson density

Lesson from $d=3$:

$$\begin{aligned} \langle \hat{q}^+(\vec{r}) \hat{q}^-(\vec{r}') \rangle &= \frac{1}{\beta} \int_0^\beta d\tau \langle \bar{q}(\vec{r}, \tau) q(\vec{r}, \tau+0^-) \rangle \\ &= \frac{1}{\beta} \int_0^\beta d\tau \langle \sqrt{\rho(\vec{r}, \tau) \rho(\vec{r}', \tau+0^-)} e^{-i\theta(\vec{r}, \tau)} e^{i\theta(\vec{r}', \tau+0^-)} \rangle \\ &\stackrel{\text{condensation}}{\sim} \rho_0 \langle e^{-i\theta(\vec{r}, \tau)} e^{i\theta(\vec{r}', \tau+0^-)} \rangle \\ &\rightarrow \text{const. for } |\vec{r} - \vec{r}'| \rightarrow \infty \end{aligned}$$

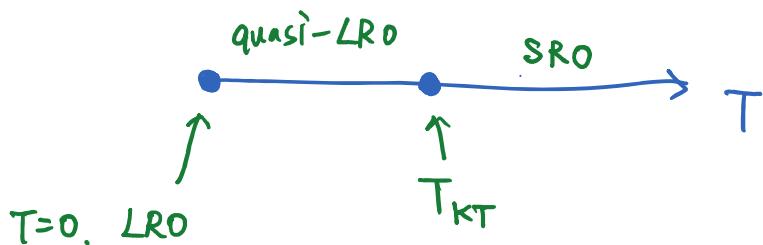
off-diagonal long-range order (ODLRO) (C.N. Yang, 1962)

Long-range phase coherence !

$d=2$: true LRO not possible,
quasi-LRO (algebraic order) still possible!

$$\text{c.f. KT story: } \langle e^{i\phi(\vec{r})} e^{-i\phi(\vec{r}')} \rangle \sim \frac{1}{|\vec{r} - \vec{r}'|^k}$$

tentative phase diagram for $d=2$ weakly interacting bosons:



Cold atom experiments: trap change dispersion relation,
BEC might still be possible, $0 < T_{\text{BEC}} < T_{\text{KT}}$

Reference on $d=2$ weakly interacting bosons:

A. Posazhennikova, Rev. Mod. Phys. 78, 111 (2006)

Possible story:

$$Z = \int D\rho D\theta e^{-S}$$

$$S = \int_0^B d\tau \int d^2\vec{r} \left\{ i\rho \partial_\tau \theta + \frac{1}{2m} \left[\underbrace{\frac{1}{2\rho} (\nabla \rho)^2 + \rho (\nabla \theta)^2}_{+ \frac{1}{2} g \rho^2} - \mu \rho \right] \right\}$$

(?) ignore if we
focus on low-energy,
long-wavelength limit.

density fluctuations are small:

$$\rho(\vec{r}, \tau) = \rho_0 + \delta\rho(\vec{r}, \tau)$$

Consider Gaussian fluctuations of $\delta\theta$ and integrate over $\delta\theta$.

Effective theory:

$$S_{\text{eff}}[\theta] \simeq \int_0^\beta d\tau \int d^2\vec{r} \left[A(\partial_\tau \theta)^2 + \frac{\rho_0}{zm} (\nabla \theta)^2 \right]$$

depending on $\frac{N}{V}, g, m, \dots$

Cannot be reliably extracted
from pure field theory calculations

$$= \int_0^\beta d\tau \int d^2\vec{r} \frac{1}{K} \left[\frac{1}{V} (\partial_\tau \theta)^2 + V(\nabla \theta)^2 \right]$$

↑

const., depending on microscopic parameters

- $d = 1$

$$S_{\text{eff}}[\theta] = \int_0^\beta d\tau \int d^2\vec{r} \frac{1}{K} \left[\frac{1}{V} (\partial_\tau \theta)^2 + V(\partial_x \theta)^2 \right]$$

famous Tomonaga-Luttinger liquid theory!

phase diagram:



δ -potential model becomes the integrable Lieb-Lininger model.

\Rightarrow V and K can be determined by exact solution (Bethe ansatz)

Then, asymptotic behaviors of correlation functions
can be calculated from field theory.

(NOT easy to calculate correlation functions in Bethe ansatz)