TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

SFB 1143

Rechenmethoden Lehramt Physik (WS22/23)

Vorlesung 10: Vektoranalysis + Linienintegral

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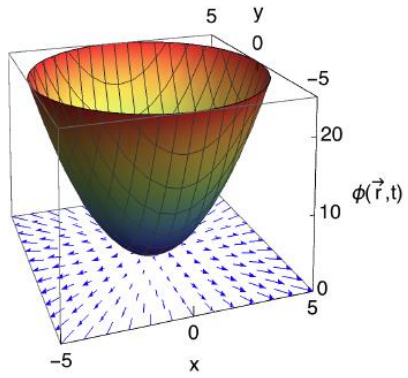
§ 6.1 Differential-Operatoren

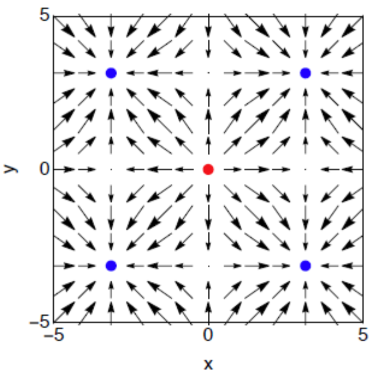
Gradient

$$\vec{a}\,\alpha = \vec{b} \quad \leftrightarrow \quad \nabla f = \vec{\nabla} f$$



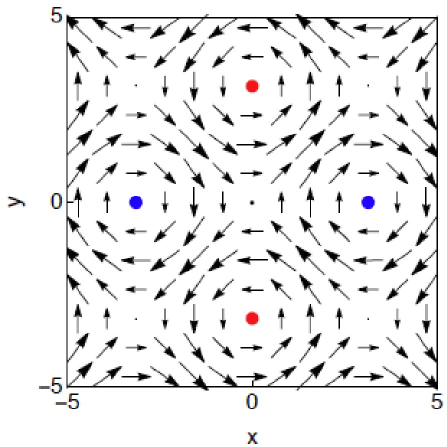
Divergenz

$$\vec{a} \cdot \vec{b} = c \quad \leftrightarrow \quad \nabla \cdot \vec{f} = \vec{\nabla} \cdot \vec{f}$$



Rotation

$$\vec{a} \times \vec{b} = \vec{c} \quad \leftrightarrow \quad \nabla \times \vec{f} = \vec{\nabla} \times \vec{f}$$



§ 6.1 Differential-Operatoren

- Laplace-Operator Δ:

$$\Delta = (\nabla \cdot \nabla) \quad \Rightarrow \quad \Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$$

Beispiele:

$$\Delta f(x,y) = \frac{\partial^2 f(\vec{x})}{\partial x^2} + \frac{\partial^2 f(\vec{x})}{\partial y^2}$$

$$\Delta \vec{A}(\vec{r},t) = \Delta \sum_{i=1}^{d} A_i(\vec{r},t)\,\vec{e}_i = \sum_{i=1}^{d} (\Delta\, A_i(\vec{r},t))\,\vec{e}_i$$

§ 6.2 Nabla-Identitäten

- Nabla-Identitäten: Eigenschaften von Levi-Civita-Tensor und Kronecker-Delta verwenden!

  - Zyklisch: $\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$

  - Antisymmetrisch: $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj}$

  - Summe (mit Einstein-Konvention): $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

§ 6.2 Nabla-Identitäten

- Rechentricks mit dem ∇-Operator:
  - div rot = 0: $\nabla \cdot [\nabla \times \vec{A}(\vec{r}, t)] = 0$
  - rot grad = 0: $\nabla \times \nabla \phi(\vec{r}, t) = 0$
  - div grad = Laplace: $\nabla \cdot \nabla \phi(\vec{r}, t) = \Delta\, \phi(\vec{r}, t)$

§ 6.2 Nabla-Identitäten

Beispiel 1: $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \Delta\vec{A}$

§ 6.2 Nabla-Identitäten

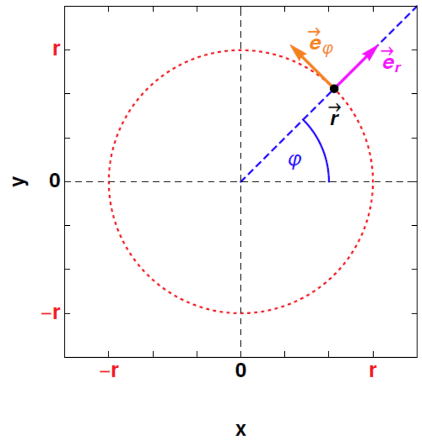
Beispiel 2: $\nabla \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\nabla \times \vec{A}\right) - \vec{A} \cdot \left(\nabla \times \vec{B}\right)$

§ 6.3 Differential-Operatoren in krummlinigen Koordinaten

- Polarkoordinaten in 2D:

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\nabla\,\phi(r,\varphi) = \begin{pmatrix} \frac{\partial \phi(r,\varphi)}{\partial x} \\ \frac{\partial \phi(r,\varphi)}{\partial y} \end{pmatrix} = ?$$



$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \rightarrow \quad \vec{r} = \begin{pmatrix} r\ \cos(\varphi) \\ r\ \sin(\varphi) \end{pmatrix}$$

$$\vec{e}_r(r,\varphi) = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$$

$$\vec{e}_\varphi(r,\varphi) = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{pmatrix}$$

§ 6.3 Differential-Operatoren in krummlinigen Koordinaten

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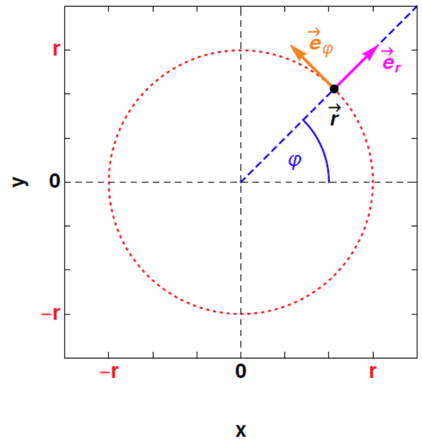
$$\nabla\,\phi(r,\varphi) = \begin{pmatrix} \frac{\partial\phi(r,\varphi)}{\partial x} \\ \frac{\partial\phi(r,\varphi)}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial\phi(r,\varphi)}{\partial r}\,\frac{\partial r}{\partial x} + \frac{\partial\phi(r,\varphi)}{\partial \varphi}\,\frac{\partial \varphi}{\partial x} \\ \frac{\partial\phi(r,\varphi)}{\partial r}\,\frac{\partial r}{\partial y} + \frac{\partial\phi(r,\varphi)}{\partial \varphi}\,\frac{\partial \varphi}{\partial y} \end{pmatrix}$$



$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \rightarrow \quad \vec{r} = \begin{pmatrix} r\,\cos(\varphi) \\ r\,\sin(\varphi) \end{pmatrix}$$

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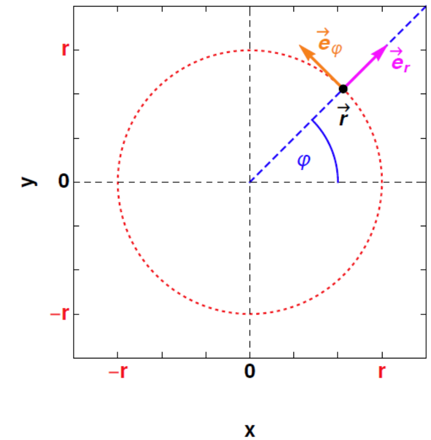
§ 6.3 Differential-Operatoren in krummlinigen Koordinaten

- Polarkoordinaten in 2D:

$$\nabla\,\phi(r,\varphi) = \begin{pmatrix} \frac{\partial\phi(r,\varphi)}{\partial x} \\ \frac{\partial\phi(r,\varphi)}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial\phi(r,\varphi)}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi(r,\varphi)}{\partial\varphi}\frac{\partial\varphi}{\partial x} \\ \frac{\partial\phi(r,\varphi)}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\phi(r,\varphi)}{\partial\varphi}\frac{\partial\varphi}{\partial y} \end{pmatrix}$$



$$\Rightarrow \quad \nabla\,\phi(r,\varphi) = \left(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\varphi\frac{1}{r}\frac{\partial}{\partial\varphi}\right)\phi(r,\varphi)$$

$$\nabla$$

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \to \quad \vec{r} = \begin{pmatrix} r\,\cos(\varphi) \\ r\,\sin(\varphi) \end{pmatrix}$$

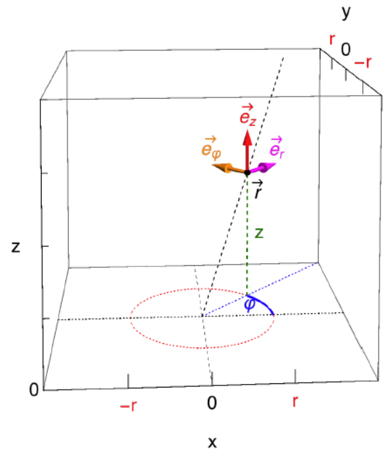
$$\vec{e}_r(r,\varphi) = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$$

$$\vec{e}_\varphi(r,\varphi) = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{pmatrix}$$

Beispiel: $g(r) = 1/r$

§ 6.3 Differential-Operatoren in krummlinigen Koordinaten

Zylinderkoordinaten in 3D:



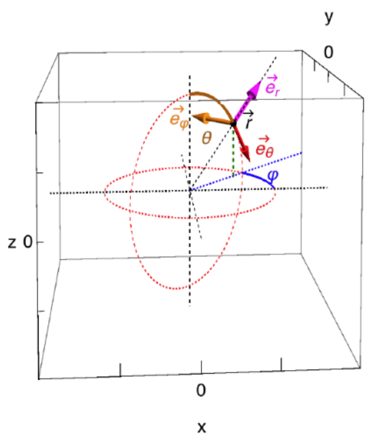
$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \rightarrow \quad \vec{r} = \begin{pmatrix} \rho\,\cos(\varphi) \\ \rho\,\sin(\varphi) \\ z \end{pmatrix}$$

$$\vec{e}_\rho(\rho,\varphi,z) = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \qquad \vec{e}_\varphi(\rho,\varphi,z) = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} \qquad \vec{e}_z(\rho,\varphi,z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \nabla = \vec{e}_\rho \frac{\partial}{\partial \rho} + \vec{e}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \vec{e}_z \frac{\partial}{\partial z}$$

§ 6.3 Differential-Operatoren in krummlinigen Koordinaten

Kugelkoordinaten in 3D:



$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \rightarrow \quad \vec{r} = \begin{pmatrix} r\,\cos(\varphi)\,\sin(\theta) \\ r\,\sin(\varphi)\,\sin(\theta) \\ r\,\cos(\theta) \end{pmatrix}$$

$$\vec{e}_r(r,\theta,\varphi) = \begin{pmatrix} \sin(\theta)\,\cos(\varphi) \\ \sin(\theta)\,\sin(\varphi) \\ \cos(\theta) \end{pmatrix} \qquad \vec{e}_\theta(r,\theta,\varphi) = \begin{pmatrix} \cos(\theta)\,\cos(\varphi) \\ \cos(\theta)\,\sin(\varphi) \\ -\sin(\theta) \end{pmatrix} \qquad \vec{e}_\varphi(r,\theta,\varphi) = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}$$

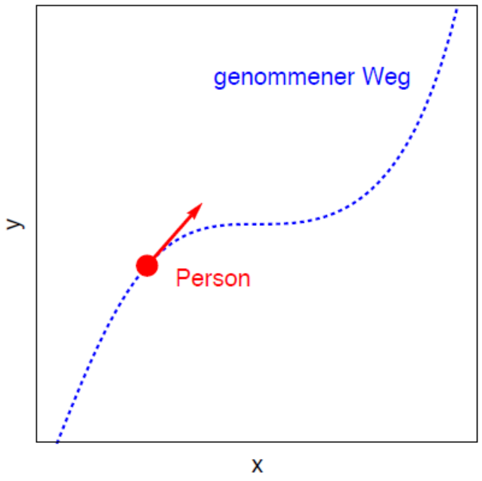
$$\Rightarrow \quad \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\varphi \frac{1}{r\,\sin(\theta)} \frac{\partial}{\partial \varphi}$$

§ 7.1 Bahnkurven

- Bewegung eines punktförmigen Teilchens durch eine Bahnkurve:

2D: $$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

3D: $$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$



Anfangspunkt: $t = t_0$ $\vec{r}(t_0)$

Endpunkt: $t = t_f$ $\vec{r}(t_f)$

§ 7.1 Bahnkurven

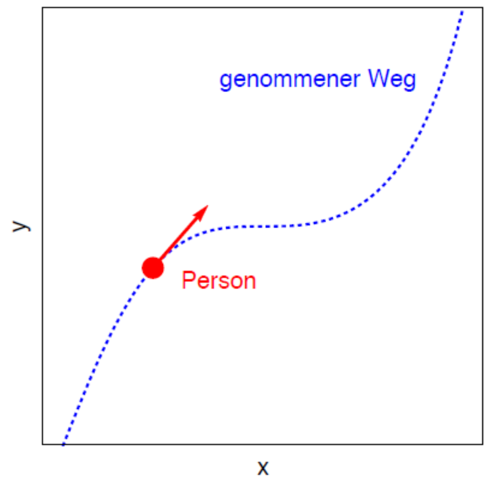
Beispiel: $\vec{r}(t) = (\cos(t), \sin(t), 0)^T,\ t \in [0, \pi]$

§ 7.1 Bahnkurven

- Geschwindigkeit und Beschleunigung:

$$\vec{v}(t) = \dot{\vec{r}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$$

$$\vec{a}(t) = \ddot{\vec{r}}(t) = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix}$$



Wie groß ist die Geschwindigkeit?

$$|\vec{v}(t)| = |\dot{\vec{r}}(t)| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$$

Richtung: $\vec{v}(t)$ steht zu jedem Zeitpunkt tangential zur Bahn $\vec{r}(t)$!

§ 7.1 Bahnkurven

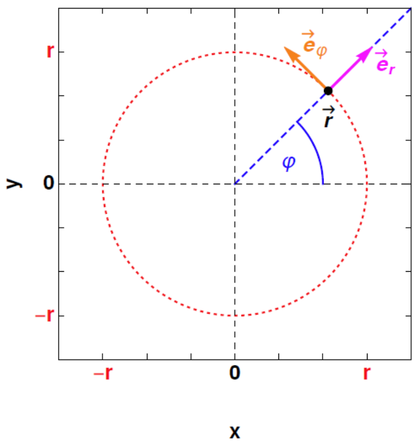
Beispiel: $\vec{r}(t) = (t, t^2)^T,\ t \in [0,1]$

$\vec{v}(t) = ?$ $\vec{a}(t) = ?$

§ 7.1 Bahnkurven

- Geschwindigkeit in Polarkoordinaten:

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} r(t)\cos\varphi(t) \\ r(t)\sin\varphi(t) \end{pmatrix}$$



$$\vec{e}_r(r,\varphi) = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$$

$$\vec{e}_\varphi(r,\varphi) = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{pmatrix}$$