Lecture Notes and Exercises on

Quantum Phase Transitions

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Literature

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Herbut, I. (2007). *A Modern Approach to Critical Phenomena*. Cambridge University Press.

Sachdev, S. (2011). *Quantum Phase Transitions*. 2nd ed. Cambridge University Press.

Vojta, M. (2015). *Thermal and Quantum Phase Transitions*. Lecture notes. URL: `https://tu-dresden.de/mn/physik/itp/ressourcen/dateien/arbeitsgruppe-quantum-critical-matter/qpt-ss24/vojta_quantum-phase-transitions.pdf`.

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Part I.
Lecture

1. Introduction

Definition 1.1 (Phase)**.** A *phase* is an equilibrium state of matter, whose qualitative characteristics do not change upon small change of external parameters. In this sense, it is a "stable" state. This definition generally justifies the assumption that the thermodynamic potential varies *analytically* when in a stable phase.

Phases are

- characterized by *symmetry* of the density operator $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ and
- seperated by *phase transitions*.

Definition 1.2 (Phase transition)**.** A *phase transition* is a point in parameter space at which equilibrium properties of a system change qualitatively, hence the system is "untable". Here, the thermodynamic potential is *nonanalytic*.

Phase transitions can

- be *continuous* or *discontinuous* and
- occur at $T > 0$ called *thermal* transitions or at $T = 0$ called *quantum* transitions.

Definition 1.3 (Quantum phase transition, QPT)**.** A *quantum phase transition* is a phase transition at $T = 0$, which occurs upon varying non-thermal control parameter such as pressure, magnetic field, chemical composition, etc. Here, the ground-state energy is *nonanalytic* in the control parameter.

Remark 1.4. A quantum phase transition is apparently driven by quantum flucutations. At $T = 0$ the state is described by a single phase-coherent (many-body) wavefunction, with flucations describing the deviations from a reference state.

Despite only occuring at zero temperature $T = 0$, the quantum phase transition is experimentally relevant, because it influences a systems properties even at finite temperature. Signatures of this "novel state of matter" are the dynamical structure factor $S(\vec{k}, \omega)$ as shown in Fig. 1 and the heat capacity. The latter scales differently in a normal state of matter than in the quantum critical region, e.g. $c_V \propto T$ for a Fermi liquid and $c_V \propto e^{-\Delta/k_B T}/T$ for a gapped magnet with gap Δ as opposed to $c_V \propto T^{d/z}$ with d the dimension d and *dynamical critical exponent* z in the quantum critical region.

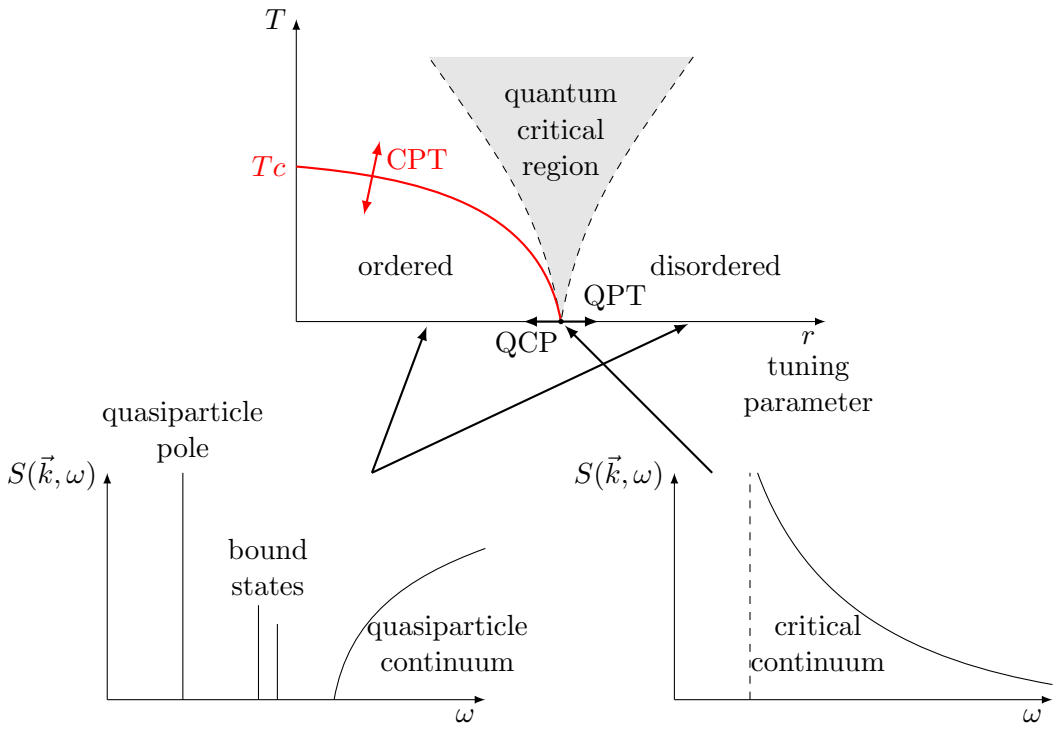
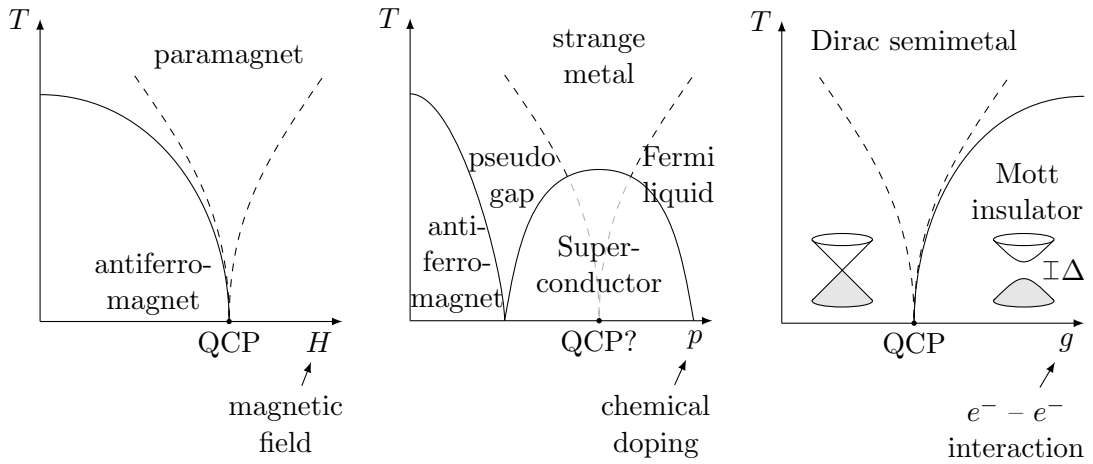


Figure 1: A quantum phase diagram with classical phase transition (CPT) and quantum phase transition (QPT) at the quantum critical point (QCP). The behavior of the dynamical structure factor $S(\vec{k},\omega)$ is qualitatively different in the ordered phases and at the QCP.



(a) Magnet in field (b) High-T_c superconductor (c) Interacting semimetal

Figure 2: Phase diagrams of different materials.

Example 1.5 (Quantum phase transitions)**.** Examples for quantum phase transitions are

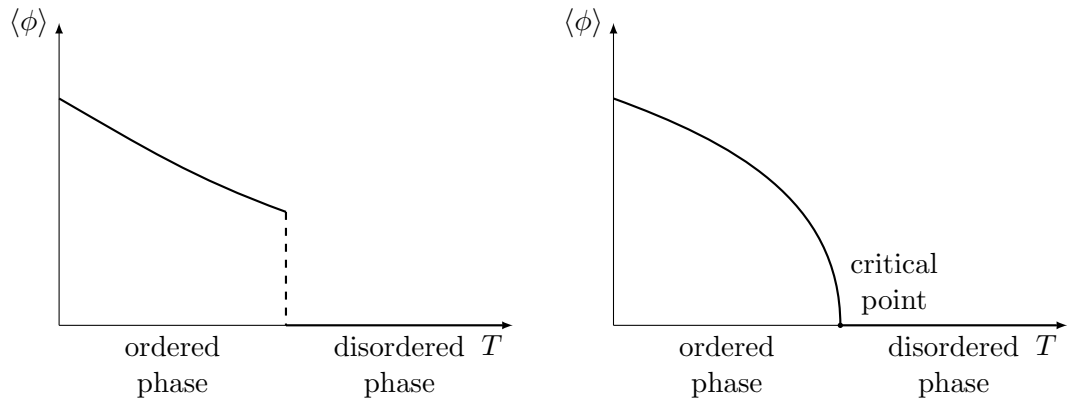
- a magnet in an external field, cf. Fig. 2a,
- a high-T_c superconductor, cf. Fig. 2b,
- an interacting semimetal, cf. Fig. 2c,
- disordered systems with "Anderson transitions",
- cold atoms on an optical lattice, or
- quark matter with "chiral symmetry breaking" in QCD.

2. Classical Phase Transitions

2.1. Definitions

Summary

- A phase transition seperates states of a medium with different characteristic properties.
- It is probed by an order parameter, which vanishes in the disordered phase and is finite in the ordered phase.
- The transition is discontinuous for first order transitions and continuous for continuous transitions.



(a) First order phase transition

(b) Second order phase transition

Figure 3: Difference of the behavior of the order parameter between a first and a second order phase transition.

- A continuous transition onsets at the critical point.

Definition 2.1 (Order parameter)**.** The *order parameter* is an observable ϕ, for which

$$\langle\phi\rangle \begin{cases} = 0 & \text{in disordered phase} \\ \neq 0 & \text{in ordered phase} \end{cases}$$

with $\langle\cdot\rangle$ the thermodynamic average ($T \neq 0$) or quantum expectation value.

Remark 2.2. The following remarks concern the order parameter:

- ϕ is usually a *local* observable $\phi = \phi(\vec{r}, t)$, with the counter-example being the volume enclosed by the Fermi surface of a metal
- ϕ is *not unique*
- ϕ is sometimes not known

Example 2.3. For a ferromagnet, one may choose $\vec{\phi}(\vec{r}_i) = \vec{S}_i$ the local magnetization at site i.

In a *first-order* transition, the order parameter changes *discontinuously* at the transition; in a *continuous transition*, the order parameter varies continuously across the transition. This is shown in Fig. 3.

The *critical point* is the transition point of a continuous transition.

To describe *correlations*, for an order parameter $\phi = \phi(\vec{r}, t)$, one defines the *correlation functions* or *two-point functions* by

$$\langle\phi(\vec{r}, t)\phi(\vec{r}', t')\rangle.$$

Definition 2.4 (Correlation length)**.** In the stable phase, the order parameter correlation function typically follows an exponential law

$$\langle\phi(\vec{r})\phi(\vec{r}')\rangle - \langle\phi(\vec{r})\rangle\langle\phi(\vec{r}')\rangle \propto e^{-\frac{|\vec{r}-\vec{r}'|}{\xi}}$$

with the *correlation length* ξ.

Remark 2.5. The correlation length ξ has the following properties:

- ξ diverges at a critical point, then

$$\langle\phi(\vec{r})\phi(\vec{r}')\rangle - \langle\phi(\vec{r})\rangle\langle\phi(\vec{r}')\rangle \propto \frac{1}{|\vec{r}-\vec{r}'|^{d-2+\eta}}$$

  where η is the *anomalous dimension*. Then correlation function becomes a *power law* and is hence scale invariant.

- Near criticality, ξ is large and becomes the only length scale characterizing the low-energy physics $a/\xi \to 0$.

Definition 2.6 (Spontaneous symmetry breaking)**.** Consider a Hamiltonian H that is invariant under a symmetry generated by S, i.e., $[H, S] = 0$. If the system's density operator ρ is not symmetric under S, i.e., $[\rho, S] \neq 0$, then the symmetry S is *spontaneously broken*.

Remark 2.7.

- Spontaneous symmetry breaking requires nonanalyticity of the thermodynamic potential, otherwise $[H, S] = 0$ would imply $[e^{-\beta H}, S] = 0$. This is satisfied for a phase transition.

- The singularity can be seen by including a small *conjugate field* h such that

$$H \mapsto H - h\phi.$$

  Then spontaneous symmetry breaking occurs if $\lim_{h\to 0^+} \rho(h) \neq \rho(h = 0)$.

- Spontaneous symmetry breaking is impossible in a finite-size system, because $e^{-\beta H}$ is not singular for $N < \infty$. For $N = \infty$ the operator $e^{-\beta H}$ may not be trace-class anymore, i.e., $\operatorname{tr} e^{-\beta H} = \infty$ such that ρ is not well-defined. Therefore, spontaneous symmetry breaking implies that

$$\lim_{h\to 0}\lim_{N\to\infty} \neq \lim_{N\to\infty}\lim_{h\to 0}.$$

- In nature, finite systems still exhibit spontaneous symmetry breaking.

– One approach to understand this is coherence. For large systems, the energy gap between the ground state and the first excited state becomes arbitrarly small. Then "stable" symmetry breaking states can be defined in the sense that their coherence time $\tau_c \propto 1/\Delta$ is much longer than any interval between reasonable measurements $\tau \ll \tau_c$. If the system is then measured in one symmetry breaking state, it remains in it.

– Another idea is that through couplings to the environment a small bias towards one state in the ground state manifold may be introduced. Any finite extent (invariant under increase of the system size) of this coupling will eventually dominate the effective Hamiltonian of the system.

2.2. Generic Phase Diagrams of Fluids and Magnets

Summary

- The liquid-gas critical point is similar to that of a magnet with order parameters $\rho \leftrightarrow M$ and external parameters $H, T \leftrightarrow \rho, T$.

In Fig. 4 the phase diagrams of liquids and magnets is compared.

2.3. Landau Theory of Phase Transitions

Summary

- In Landau theory, the order parameter is treated as a parameter in the free energy of the system.
- The free energy is then expanded to include all symmetry allowed terms.
- In Ginzburg-Landau theory, the order parameter is dependent on space $\phi \to \phi(\vec{r})$ and spatial variations are included in the free energy functional. Here, the propagator $G = \langle\phi\phi\rangle - \langle\phi\rangle^2$ measures fluctuations around $\langle\phi\rangle$.
- Below the upper critical dimension $d < d_c^+$ the fluctuations diverge as $\langle\delta\phi\delta\phi\rangle/\langle\phi\rangle^2 \to \infty$.
- Above the upper critical dimension other length scales become relevant.

We assume that the phase transition is uniquely described in terms of a local order parameter ϕ. The idea is then to express a *generalized thermodynamic potential* f, the *Landau functional*, in terms of ϕ.

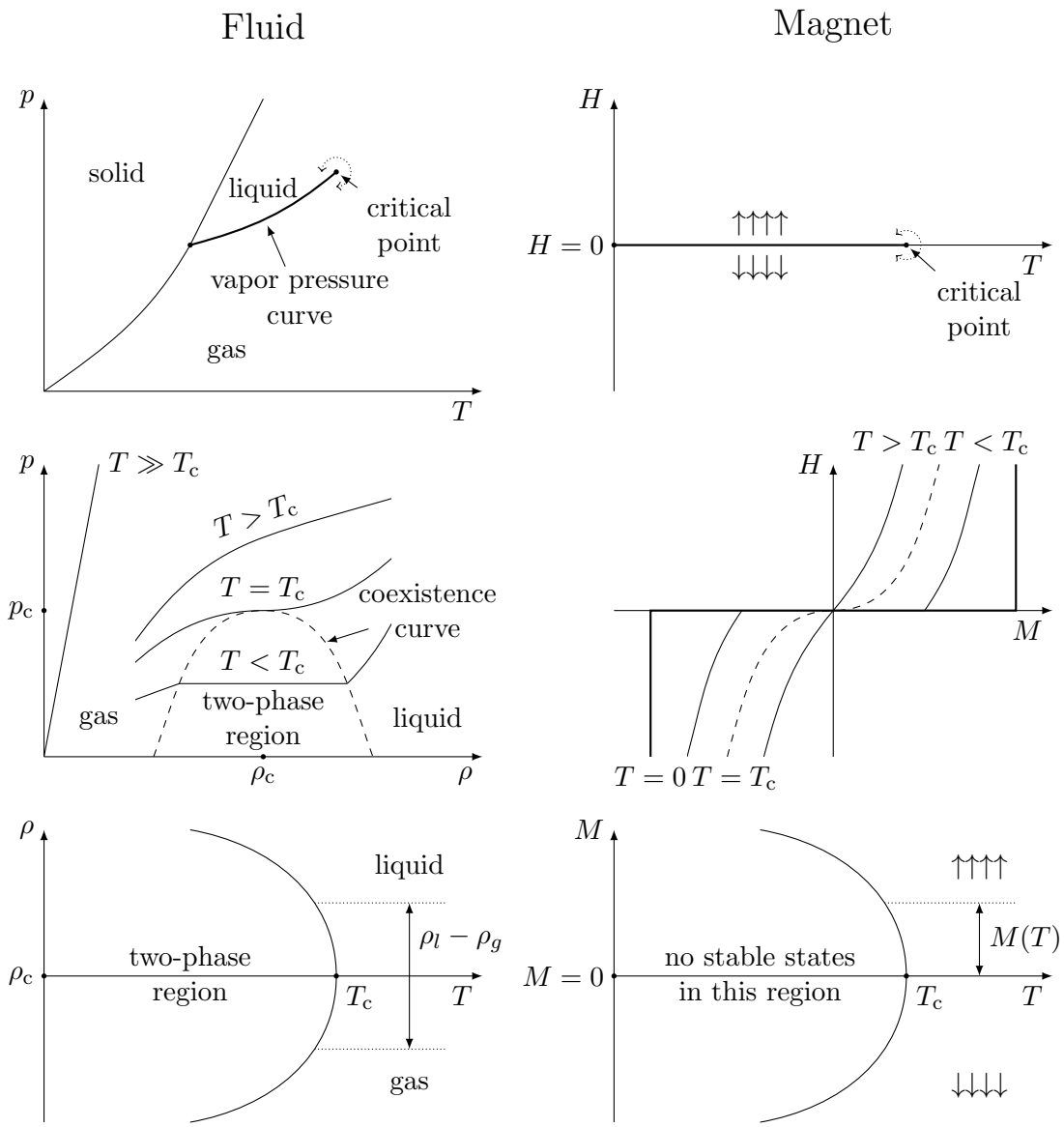


Figure 4: Comparison of generic phase diagrams for liquids with pressure p, temperature T and density ρ and magnets with magnetic field H, temperature T and magnetization M; their remarkable similarity becomes apparent.

Example 2.8 (Ferromagnet)**.** Specifying the magnetic field H and the temperature T fixes the magnetization $\phi = M$. The generalized potential is then

$$f(T, H) \mapsto f(T, H, \phi)$$

with $f(T, H)$ the thermodynamic potential and $f(T, H, \phi)$ the Landau functional. We assume that $f(T, H, \phi)$ is non-singular. The equilibrium state is obtained from

$$\left.\frac{\partial f(T, H, \phi)}{\partial \phi}\right|_{\phi=\phi_{\text{eq}}} = 0$$

and $f(T, H) = f(T, H, \phi_{\text{eq}}(T, H))$.

One can expand the Landau potential (near criticality) as

$$f(T, H) = f_n + f_0 \left(\frac{a(T)}{2}\phi^2 + \frac{b(T)}{4}\phi^4 + \frac{c(T)}{6}\phi^6 + \ldots - h\phi \right)$$

where $h = H/f_0$ is the magnetic field (called *conjugate* to the order parameter) and f_n is only weakly T dependent.

Remark 2.9.

- f includes all *symmetry-allowed terms*, e.g., for $H = 0$ the Ising symmetry

  $$\mathbb{Z}_2 : \phi \mapsto -\phi$$

  implies that only even power in ϕ are allowed.
- For vector order parameters $\vec{\phi}$, the rotational symmetry

  $$\mathrm{O}(N) : \vec{\phi} \mapsto R\vec{\phi}$$

  implies that only $(\vec{\phi}^2) = \vec{\phi} \cdot \vec{\phi}$ and higher order terms $(\vec{\phi}^2)^2, (\vec{\phi}^2)^4$ are allowed.
- The coefficients $a(T), b(T), \ldots$ are smooth functions of external parameters that preserve the symmetry, in particular, of the temperature.

Discussion for $h = 0$ and $b > 0$. We have

$$f - f_n = \frac{a}{2}\phi^2 + \frac{b}{4}\phi^4,$$

as shown in Fig. 5. If $b(T) > 0$ for $T \approx T_{\text{c}}$, then we can neglect higher-order terms $\propto \mathcal{O}(\phi^6)$ for ϕ near ϕ_{eq}, which we obtain as

$$\left.\frac{\partial f}{\partial \phi}\right|_{\phi_{\text{eq}}} = 0, \quad \left.\frac{\partial^2 f}{\partial \phi^2}\right|_{\phi_{\text{eq}}} > 0 \quad \Longrightarrow \quad \phi_{\text{eq}} = \begin{cases} 0, & a > 0 \text{ (disordered state)}, \\ \pm\sqrt{\frac{-a}{b}}, & a < 0 \text{ (ordered state)}. \end{cases}$$

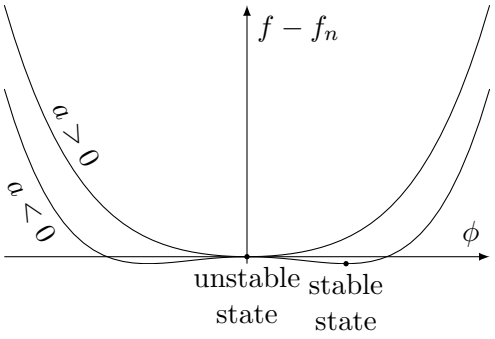


Figure 5: Illustration of $f - f_n = \frac{a}{2}\phi^2 + \frac{b}{4}\phi^4$ for $b > 0$ and different a.

The phase transition occurs at $T = T_\mathrm{c}$, when $a(T) = 0$. Hence, we may expand a in the reduced temperature $t = \frac{T - T_\mathrm{c}}{T_\mathrm{c}}$ as

$$a(T) = \alpha t + \mathcal{O}(t^2)$$

where we assumed that $a(T)$ is smooth across the transition. Then the order parameter is

$$\langle\phi\rangle(T) = \begin{cases} 0, & T > T_\mathrm{c}, \\ \pm\sqrt{\frac{\alpha}{b}(-t)}, & T < T_\mathrm{c}. \end{cases}$$

Remark 2.10.

- In general $\langle\phi\rangle(t < 0) \propto (-t)^\beta$ with a critical exponent β. In Landau theory we have $\beta = \frac{1}{2}$.
- For the spontaneously broken $\mathbb{Z}_2$, we have $\langle\phi\rangle(t < 0) = \pm\sqrt{\frac{-a}{b}}$.
- For the spontaneously broken $\mathrm{O}(N)$, we have $\langle\phi\rangle(t < 0) = \sqrt{\frac{-a}{b}}\vec{e}_0$ with $\vec{e}_0$ an arbitrary direction.

We find the following thermodynamic observables:

$$f(T) = f(T, \phi_\mathrm{eq}) = \begin{cases} f_n(T), & T > T_\mathrm{c} \\ f_n(T) - f_0 \frac{\alpha^2}{4b}\left(\frac{T - T_\mathrm{c}}{T_\mathrm{c}}\right)^2, & T < T_\mathrm{c} \end{cases}$$

$$\frac{S}{V} = -\left(\frac{\partial f}{\partial T}\right)_V = \begin{cases} s_0(T), & T > T_\mathrm{c} \\ s_0(T) - f_0 \frac{\alpha^2}{2b}\frac{T_\mathrm{c} - T}{T_\mathrm{c}^2}, & T < T_\mathrm{c} \end{cases}$$

$$c_v = \frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_T = \begin{cases} c_0, & T > T_\mathrm{c} \\ c_0 + f_0 \frac{\alpha^2}{2b}\frac{T}{T_0^2}, & T < T_\mathrm{c} \end{cases}$$

In general, $c_v(T) = c_\pm |t|^{-\alpha} + \mathcal{O}(t^2)$ with critical exponent α ($\alpha = 0$ in Landau theory), as shown in Fig. 6.

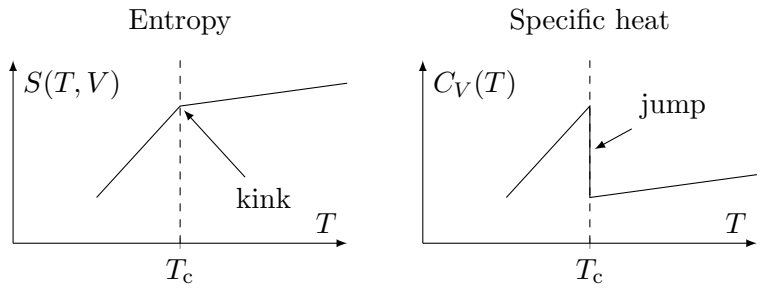


Figure 6: Behavior of entropy and specific heat in $\mathbb{Z}_2$ Landau theory near the critical temperature T_c.

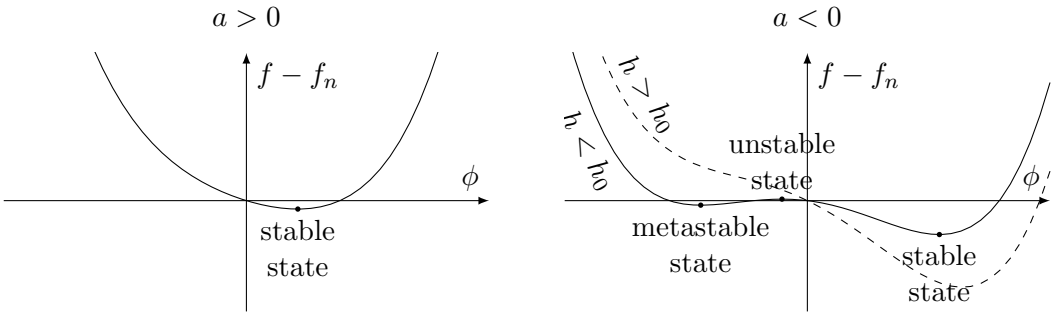


Figure 7: Illustration of $f - f_n = \frac{a}{2}\phi^2 + \frac{b}{4}\phi^4 - h\phi$ for $b > 0$ and different a.

Discussion for $h \neq 0$ and $b > 0$. The equilibrium state obtained from $\frac{\partial f}{\partial \phi} = 0$ is given by the implicit equation

$$a\phi + b\phi^3 = h$$

with $a = a(T) = \alpha t + \mathcal{O}(t^2)$. For $a > 0$ this has one solution, while for $a < 0$ either three solutions for $h < h_0$ or one solution for $h > h_0$ exist, see Fig. 7.

We can define the *susceptibility* χ as

$$\phi(h) = \phi_\mathrm{spont} + \chi(T)h + \mathcal{O}(h^2)$$

with $\chi = \partial\phi/\partial h|_{\phi_\mathrm{spont}}$ the *order parameter susceptibility* and ϕ_spont the zero-field order parameter.

For small h, we obtain from

$$a(T)[\phi_\mathrm{spont} + \chi(T)h] + b[\phi_\mathrm{spont} + \chi(T)h]^3 = h, \qquad \phi_\mathrm{spont} = \begin{cases} 0, & T > T_\mathrm{c}, \\ \sqrt{\frac{-a}{b}}, & T < T_\mathrm{c}, \end{cases}$$

that

$$\chi(T) = \begin{cases} \frac{1}{\alpha t}, & T > T_\mathrm{c}, \\ \frac{1}{2\alpha|t|}, & T < T_\mathrm{c}, \end{cases}$$

with $\lim_{T\to T_c} \chi(T) = \infty$, see Fig. 8.

Proof. We expand the cubic to obtain (labeling $\phi = \phi_{\text{spont}}$) as

$$(a + b\phi^2)\phi + (a\chi + 3b\phi^2\chi - 1)h + \mathcal{O}(h^2) = 0.$$

The first term yields the spontaneous magnetization as $\phi = \sqrt{-a/b}$, while the second one yields the susceptibility with $a = \alpha t$

$$\chi = \frac{1}{a + 3b\phi^2} = \begin{cases} \frac{1}{\alpha t}, & \phi^2 = 0, \\ \frac{1}{-2\alpha t}, & \phi^2 = -\frac{a}{b}, \end{cases}$$

in the disordered and ordered phase respectively. □

In general $\chi(T) \propto |t|^{-\gamma} + \mathcal{O}(t^2)$ with critical exponent γ; in Landau theory $\gamma = 1$.

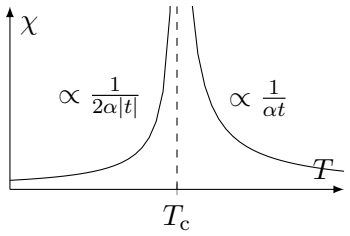


Figure 8: Behavior of the order parameter susceptibility χ near the critical point.

At criticality ($T = T_c$, $a(T) = 0$), we obtain the *critical isotherm* for finite h as

$$\phi = \left(\frac{h}{b}\right)^{1/3}.$$

In general $\phi \propto |h|^{\frac{1}{\delta}}$ with critical exponent δ; in Landau theory $\delta = 3$.

The phase-diagrams are depicted in Fig. 9.

Remark 2.11.

- The line of first-order transition at $h = 0$ for $T < T_c$ is (a).
- Here, the two states at $h = 0^+$ and $h = 0^-$ are related by $\mathbb{Z}_2$ symmetry. Therefore, the latent heat $Q = \Delta S = 0$.
- Metastable states for $h < h_0$ and $T < T_c$ result in a hysteresis near $h \approx 0$.
- Continuous transition at $T = T_c$ and $h = 0$ in (c, d).
- "Crossover" for $T > T_c$ (b) or $h \neq 0$ (e).

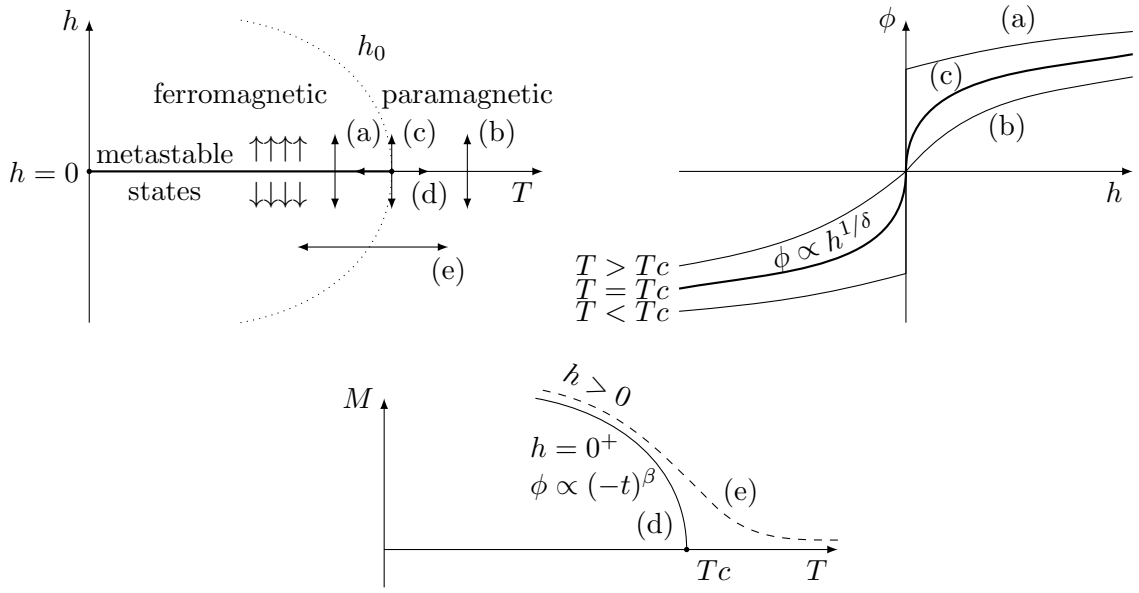


Figure 9: Phase diagrams for $\mathbb{Z}_2$ Landau theory with non-vanishing external field.

Ginzburg-Landau Theory and Spatial Correlations. Now, we allow for spatial variations of the order-parameter or fluctuations

$$\phi \to \phi(\vec{r})$$

where $\vec{r}$ is a continuous space coordinate and $\phi(\vec{r})$ is a smooth function, capturing variations on length scales much larger than microscopic lengths (in the so-called *continuum limit description*). The *Ginzburg-Landau functional* is then

$$f(T, H, \phi(\vec{r})) = f_n + f_0 \left[\frac{a}{2}\phi(\vec{r})^2 + \frac{b}{4}\phi(\vec{r})^4 + \xi_0 (\nabla\phi(\vec{r}))^2 - \phi(\vec{r})h \right] + \mathcal{O}(\phi^6, \nabla^4, \nabla^2\phi^4).$$

Here, ξ_0 can be thought of as an energy cost of spatial variations. We retrieve the Ginzburg-Landau (free) energy F as

$$F = \int \mathrm{d}^d\vec{r} f(\vec{r}).$$

The correlations are defined with the *propagator* G as

$$G(\vec{r}, \vec{r}') = \langle \phi(\vec{r})\phi(\vec{r}') \rangle - \langle \phi(\vec{r}) \rangle \langle \phi(\vec{r}') \rangle = \langle \phi(\vec{r})\phi(\vec{r}') \rangle - \phi_0^2$$

with $\phi_0 \equiv \langle \phi(\vec{r}) \rangle$ the homogenous order-parameter.

In the large-distance limit for $T < T_\mathrm{c}$, i.e., in the ordered phase, we have long-range order such that

$$\lim_{|\vec{r}-\vec{r}'|\to\infty} \langle \phi(\vec{r})\phi(\vec{r}') \rangle = \phi_0^2 \neq 0$$

and thus $G(\vec{r},\vec{r}')$ measures *correlations of fluctuations* around ϕ_0.

We use this to obtain the critical exponent ν by first expanding ϕ around ϕ_0 as $\phi(\vec{r}) = \phi_0 + \delta\phi(\vec{r})$. The free energy is $F(\phi(\vec{r})) = F(\phi_0) + \delta F$ with

$$\delta F = \frac{f_0}{2}\int \mathrm{d}^d\vec{r}\delta\phi(\vec{r})(\alpha t + 3b\phi_0^2 - \xi_0^2\nabla^2)\delta\phi(\vec{r})$$

with $a(T) = \alpha t + \mathcal{O}(t^2)$ and the terms linear in $\delta\phi$ vanishing due to the equilibrium condition $\frac{\delta F}{\delta\phi}|_{\phi_0} = 0$. The Fourier decomposition

$$\delta\phi(\vec{r}) = \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\delta\phi(\vec{k})e^{i\vec{k}\vec{r}}$$

yields

$$\delta F = \frac{k_B T}{2}\int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\delta\phi(-\vec{k})G^{-1}(\vec{k})\delta\phi(\vec{k})$$

with

$$G(\vec{k}) = \langle\delta\phi(\vec{k})\delta\phi(-\vec{k})\rangle = \frac{k_B T}{f_0(A(T) + \xi_0^2\vec{k}^2)},$$
$$A(T) \equiv \alpha t + 3b\phi_0^2 = \begin{cases}\alpha t, & T > T_\mathrm{c}\\ \alpha t + 3b\phi_0^2, & T < T_\mathrm{c}\end{cases}.$$

Upon Fourier back transform, one finds the spatial propagator

$$G(\vec{r},\vec{r}') = \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}G(\vec{k})e^{i\vec{k}(\vec{r}-\vec{r}')} \propto \frac{e^{-\frac{r}{\xi(T)}}}{r^{\frac{d-1}{2}}}\xi(T)^{\frac{3-d}{2}}$$

with $r \equiv |\vec{r} - \vec{r}'|$ and the correlation-length

$$\xi(T) = \begin{cases}\frac{\xi_0}{\sqrt{\alpha t}}, & T > T_\mathrm{c}\\ \frac{\xi_0}{\sqrt{2\alpha|t|}}, & T < T_\mathrm{c}\end{cases},$$

which diverges for $T \to T_\mathrm{c}$.

In general, we obtain $\xi(T) \propto |t|^{-\nu}$ with critical exponent ν. In Ginzburg-Landau theory $\nu = \frac{1}{2}$.

Remark 2.12.
- A divergent length scale ξ implies locally ordered "islands" of increasing size.
- That $\phi(\vec{r})$ varies slowly near criticality justifies the gradient expansion a posteriori.
- At criticality the correlation time $\tau_\mathrm{c} \to \infty$ leading to *critical slowing down*.

Here, at criticality with $A(T) = 0$, we have $G(\vec{k}) \propto \vec{k}^{-2} \implies G(r) \propto r^{2-d}$. In general $G(r) \propto r^{2-d-\eta}$ with *anomalous dimension* η; $\eta = 0$ in Ginzburg-Landau theory.

Fluctuations and Ginzburg criterion. Landau theory can be "derived" as a saddle-point solution of a field theory dormulated as a functional integral, while neglecting fluctuations; hence leading to a *mean-field approximation*.

The Ginzburg criterion can check for the validity of Landau theory by considering the effective relative size of fluctuations

$$\frac{\langle \delta\phi(\vec{r}=\xi(T)\vec{e})\,\delta\phi(0)\rangle}{\phi_0^2} \propto \frac{\xi(T)^{2-d}}{|t|} \propto |t|^{\frac{d-4}{2}} \xrightarrow[|t|\to 0]{} \begin{cases} 0, & d>4, \\ \infty, & d<4, \end{cases}$$

using $G(\vec{r},\vec{r}') \propto r^{2-d}$, $\phi_0 \propto |t|^{1/2}$ and $\xi \propto |t|^{-1/2}$ in Landau theory. Therefore, Landau theory is asymptotically exact for $d > 4$.

Remark 2.13.

- $d_\mathrm{c}^+ = 4$ is called the *upper critical dimension*.
- d_c^+ depends on (some general) system properties.
- Critical exponents in Landau theory called *mean-field exponents* are exact for $d > d_\mathrm{c}^+$.
- There are logarithmic corrections directly at $d = d_\mathrm{c}^+$.
- Landau theory fails *at criticality* for $d < d_\mathrm{c}^+$.
- The mean-field exponents are still observable in systems in which the numerical prefactor in the Ginzburg criterion is small, e.g., in conventional superconductors with $t_\mathrm{cr} \approx 10^{-10} \ll 1$.
- A consistent account for fluctuations beyond mean field is obtained from the renormalization group.

2.4. Critical Exponents and Universality

Summary

- At criticality, observables follow power laws of the system parameters.
- Critical exponents define universality classes for microscopically different systems.

At criticality $\xi \to \infty$ and hence there are fluctuations on all length scales, i.e., emerging *scale invariance*. There, observables follow *power laws* such as $A \propto x^y$ with A the observable, x some system parameter and y the critical exponent. Examples are listed in Table 1.

Remark 2.14.

- The critical exponent η is called the *anomalous dimension*.

Observable	Exponent	Definition	Conditions	MF value (GL) $d > d_c$	Example Ising ($d = 3$)
Specific heat	α	$C \propto \lvert t\rvert^{-\alpha}$	$t \to 0,\ h = 0$	$\alpha = 0$	$\alpha \simeq 0.110$
Order parameter	β	$\phi \propto (-t)^{\beta}$	$t \nearrow 0,\ h = 0$	$\beta = \frac{1}{2}$	$\beta \simeq 0.326$
Susceptibility	γ	$\chi \propto \lvert t\rvert^{-\gamma}$	$t \to 0,\ h = 0$	$\gamma = 1$	$\gamma \simeq 1.237$
Critical isotherm	δ	$h \propto \lvert\phi\rvert^{\delta} \operatorname{sgn}\phi$	$t = 0,\ h \to 0$	$\delta = 3$	$\delta \simeq 4.789$
Correlation length	ν	$\xi \propto \lvert t\rvert^{-\nu}$	$t \to 0,\ h = 0$	$\nu = \frac{1}{2}$	$\nu \simeq 0.630$
Correlation function	η	$G(r) \propto \lvert r\rvert^{-d+2-\eta}$	$t = 0,\ h = 0$	$\eta = 0$	$\eta \simeq 0.0363$
Correlation time	z	$\tau_c \propto \xi^z \propto \lvert t\rvert^{-\eta z}$	$t \to 0,\ h = 0$	–	–

Table 1: Common critical exponents

- The critical exponent z is called the *dynamical exponent*. It will play a special role in quantum phase transitions.
- Power laws are scale invariant, e.g.,

$$C \propto |t|^{-\alpha} \propto \xi^{\frac{\alpha}{\nu}} \overset{\xi \mapsto b\xi}{\longmapsto} b^{\frac{\alpha}{\nu}} C$$

Remark 2.15. Some comments on universality.

- Critical exponents can be are *identical* for microscopically completely different systems.
- Systems fall into *universality classes*, e.g., $\mathbb{Z}_2$ universality containing the Ising model and the liquid-gas critical point.
- Universality classes are characterized by only very few *general properties*, such as *dimension* d, *symmetry* of the order parameter, presence or absence of long-range interactions.
- The phenomenological reason is that $\xi \to \infty$ and therefore microscopic details become *irrelevant*.

2.5. Scaling Hypothesis

Summary

- Below the upper critical dimension, ξ is the only relevant length scale near criticality.
- The free energy density is homogenous in its parameters

$$f_s(t, h) = b^{-d} f_s(b^{y_t} t, b^{y_h} h).$$

- In this regime, hyperscaling relations (Josephson's law, Fisher's law, Rushbrooke's law and Griffith's law) hold.

We assume that ξ is the only length scale near criticality. Then, we may perform the scaling transformation on

$$\begin{aligned} x &\mapsto bx && \text{length,} \\ t &\mapsto b^{y_t} t && \text{reduced temperature,} \\ h &\mapsto b^{y_h} h && \text{external field.} \end{aligned}$$

The scale invariance now requires that a change in length can be compensated by a change in t and h for appropriate y_t and y_h.

The scaling hypothesis for the free energy density is then

$$f_s(t,h) = b^{-d} f_s(b^{y_t} t, b^{y_h} h)$$

with f_s the singular part of the free energy. We say that f_s is *homogenous* in both t and h. The consequence of the scaling hypothesis is that in static critical phenomena only two independent exponents, namely y_t and y_h, exist and all other may be related to them.

Example 2.16 (Correlation length)**.** We perform the scaling transformation

$$\xi \propto |t|^{-\nu} \xmapsto{t \to b^{y_t} t} b^{-y_t \nu} |t|^{-\nu} \propto b^{-y_t \nu} \xi \xmapsto{x \to bx} b^{1-y_t \nu} \xi.$$

From scale invariance it follows that $\xi \mapsto \xi$ and hence $\nu = 1/y_t$.

Example 2.17 (Free energy density at $h = 0$)**.** With no conjugate field, we have

$$f_s(t, h=0) = b^{-d} f_s(b^{y_t} t, 0).$$

Now, choose b such that $b^{y_t} |t| = 1$, i.e., $b^{-d} = |t|^{\frac{d}{y_t}} = |t|^{d\nu}$, then

$$f_s(t, h=0) = |t|^{d\nu} f_s(\pm 1, 0) \implies f_s(t, h=0) \propto |t|^{d\nu}.$$

Example 2.18 (Specific heat)**.** We can obtain *Josephson's law* from the specific heat as

$$c_V \propto T \frac{\partial^2 f}{\partial T^2} \propto |t|^{d\nu - 2} \implies \alpha = 2 - d\nu.$$

Example 2.19 (Susceptibility)**.** The correlation function at $h = 0$ scales as

$$G(r; t, h=0) \propto \frac{f_G\left(\frac{r}{\xi(t)}\right)}{r^{d-2+\eta}},$$

which at the critical point turns into $G(r; t = 0, h = 0) \propto r^{2-d-\eta}$. The susceptibility is accordingly

$$\chi = \frac{\partial \langle \phi \rangle}{\partial h}\Big|_{h\to 0} \propto \int \mathrm{d}^d \vec{r} G(r;t) \propto \int \mathrm{d}r f_G(r\,|t|^\nu) r^{1-\eta} = \underbrace{\int \mathrm{d}x f_G(x) x^{1-\eta}}_{\text{const.}} |t|^{-(2-\eta)\nu},$$

having substituted $x = r\,|t|^\nu$ and $\mathrm{d}x = |t|^\nu\,\mathrm{d}r$. From this, *Fisher's law* follows as

$$\gamma = (2-\eta)\nu.$$

Josephson's law	$\alpha = 2 - d\nu$
Fisher's law	$\gamma = (2-\eta)\nu$
Rushbrooke's law	$\alpha + 2\beta + \gamma = 2$
Griffith's law	$\alpha + \beta(\delta + 1) = 2$

Remark 2.20.

- The scaling hypothesis is an assumption that can be justified with RG.
- It holds for $d < d_\mathrm{c}^+$, but not for $d > d_\mathrm{c}^+$, in which the *hyperscaling violation* holds. Here, additional length scales become relevant.
- Mean-field exponents violate Josephson's law.

3. Statistical Mechanics and Path Integrals

3.1. Coherent-State Path Integral

Summary

- Coherent states are a prototypical example of bosonic quantum systems, which allow to describe the groundstate of a BEC.
- By expanding the partition function, a field-theoretical approach in terms of a functional integral can be derived.
- At criticality, only the $\omega_0 = 0$ mode contributes to the critical properties, as the correlation time becomes greater than then finite extent of the system $\tau_\mathrm{c} \gg \beta$ and all other modes are gapped and therefore non-critical, otherwise the gap would be a relevant length scale.
- At $T = 0$, a continuum of small-ω modes contributes because the gap vanishes.

As a prototypical example, we consider a system of *many* interacting bosons.

The Fock basis of the many-body Hilbert space is

$$|n_1, \ldots, n_N\rangle = \prod_{\alpha=1}^{N} \frac{(a_\alpha^\dagger)^{n_\alpha}}{\sqrt{n_\alpha!}}|0\rangle,$$

where $|0\rangle = |0, \ldots, 0\rangle$ is the many-body vacuum state and $a_\alpha^\dagger$ (a_α) creates (annihilates) a boson in the α-th single-particle state. They obey the commutation relation $[a_\alpha, a_{\alpha'}^\dagger] = \delta_{\alpha,\alpha'}$ and $[a_\alpha, a_{\alpha'}] = [a_\alpha^\dagger, a_{\alpha'}^\dagger] = 0$. Any state may be expanded in this basis as

$$|\Phi\rangle = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \Phi_{n_1,\ldots,n_N}|n_1, \ldots, n_N\rangle.$$

Definition 3.1 (Coherent state)**.** A *coherent state* is defined as the eigenstate of an annihilator as

$$a_\alpha|\Phi\rangle = \Phi_\alpha|\Phi\rangle$$

for all $\alpha = 1, \ldots, N$ where $\Phi_\alpha \in \mathbb{C}$.

The expansion coefficients of a coherent state are

$$\Phi_{n_1,\ldots,n_N} = \prod_{\alpha=1}^{N} \frac{(\Phi_\alpha)^{n_\alpha}}{\sqrt{n_\alpha!}},$$

such that

$$|\Phi\rangle = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \prod_{\alpha=1}^{N} \frac{(\Phi_\alpha a_\alpha^\dagger)^{n_\alpha}}{n_\alpha!}|0\rangle = e^{\sum_\alpha \Phi_\alpha a_\alpha^\dagger}|0\rangle.$$

Remark 3.2.

- The coherent state $|\Phi\rangle$ as defined above can easily be shown to satisfy $a_\alpha|\Phi\rangle = \Phi_\alpha|\Phi\rangle$ by using $[a_\alpha, (a_{\alpha'}^\dagger)^n] = n(a_\alpha^\dagger)^{n-1}\delta_{\alpha\alpha'}$.
- The coherent state $|\Phi\rangle$ is a superposition of states with arbitrary number of particles.

The bra version $\langle\Phi| = \langle 0|e^{\sum_\alpha \Phi_\alpha^* a_\alpha}$ is the left eigenstate of the creation operator with

$$\langle\Phi|a_\alpha^\dagger = \langle\Phi|\Phi_\alpha^*.$$

For the ket version, we have $a_\alpha^\dagger|\Phi\rangle = \frac{\partial}{\partial\Phi_\alpha}|\Phi\rangle$. Two coherent states $|\Phi\rangle$ and $|\Phi'\rangle$ are not orthogonal as

$$\langle\Phi|\Phi'\rangle = e^{\sum_\alpha \Phi_\alpha^*\Phi_{\alpha'}} \neq 0.$$

Proof. This can be shown by explicitly plugging in $|\Phi'\rangle$ and using $\langle\Phi|a_\alpha^\dagger = \langle\Phi|\Phi_\alpha^*$, yielding

$$\langle\Phi|\sum_{n_\alpha}\prod_{\alpha=1}^{N}\frac{(\Phi'_\alpha a_\alpha^\dagger)^{n_\alpha}}{n_\alpha!}|0\rangle = \sum_{n_\alpha}\prod_\alpha\frac{(\Phi'_\alpha\Phi_\alpha^*)^{n_\alpha}}{n_\alpha!} = \prod_\alpha\sum_{n_\alpha}\frac{(\Phi'_\alpha\Phi_\alpha^*)^{n_\alpha}}{n_\alpha!} = e^{\sum_\alpha \Phi_\alpha^*\Phi'_\alpha}.$$

□

The resolution of identity for coherent states is

$$I = \int\prod_\alpha\frac{\mathrm{d}\Phi_\alpha^*\mathrm{d}\Phi_\alpha}{2\pi i}e^{-\sum_\alpha\Phi_\alpha^*\Phi_\alpha}|\Phi\rangle\langle\Phi|,$$

therefore coherent states form an overcomplete "basis", as they are not mutually orthogonal but still any state can be expanded in terms of them.

Proof. To show that this is indeed a resolution of the identity, consider the following

$$\begin{aligned}\int\frac{\mathrm{d}\Phi^*\mathrm{d}\Phi}{2\pi i}e^{-|\Phi|^2}|\Phi\rangle\langle\Phi| &= \int_0^\infty\frac{r\mathrm{d}r}{\pi}\int_0^{2\pi}\mathrm{d}\theta e^{-r^2}\sum_{m,n=0}^{\infty}\frac{(re^{i\theta})^m}{\sqrt{m!}}|m\rangle\frac{(re^{i\theta})^n}{\sqrt{n!}}\langle n| \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^\infty\mathrm{d}r2re^{-r^2}r^{2n}|n\rangle\langle n| \\ &= \sum_n|n\rangle\langle n| = I.\end{aligned}$$

□

Partition function. The grand-canonical partition function is

$$Z = \operatorname{tr}e^{-\beta(H-\mu N)},$$

where $\beta = \frac{1}{k_BT}$ is the inverse temperature and μ the chemical potential. A generic Hamiltonian is

$$H = \sum_\alpha e_\alpha a_\alpha^\dagger a_\alpha + \sum_{\alpha,\beta,\gamma,\delta}\langle\alpha\beta|V|\gamma\delta\rangle a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma,$$

with e_α the energy eigenvalues for $V = 0$ and V a two-body interaction. The particle number operator is $N = \sum_\alpha a_\alpha^\dagger a_\alpha$. In the coherent state framework, the partition function reads

$$Z = \int\prod_\alpha\frac{\mathrm{d}\Phi_\alpha^*\mathrm{d}\Phi_\alpha}{2\pi i}e^{-\sum_\alpha\Phi_\alpha^*\Phi_\alpha}\langle\Phi|e^{-\beta(H-\mu N)}|\Phi\rangle,$$

which we rewrite by partitioning $\beta = \epsilon M$ for $M \gg 1$, such that $e^{-\beta(H-\mu N)} = \prod^M e^{-\epsilon(H-\mu N)}$, and inserting identities between the product as

$$Z = \int \prod_{k=0}^{M-1} \prod_{\alpha} \frac{\mathrm{d}\Phi^*_{\alpha,k}\mathrm{d}\Phi_{\alpha,k}}{2\pi i} e^{-\sum_{k=0}^{M-1}\sum_\alpha \Phi^*_{\alpha,k}\Phi_{\alpha,k}} \prod_{k=0}^{M-1} \langle \Phi_k | e^{-\epsilon(H-\mu N)} | \Phi_{k+1}\rangle$$

where $\Phi_0 = \Phi_M = \Phi$. For normal ordered operators $A(a^\dagger_\alpha, a_\alpha)$ as normal-ordered functions of $a^\dagger_\alpha$ and a_α, where all $a^\dagger_\alpha$ are left of all a_α, we have

$$\langle \Phi | A(a^\dagger_\alpha, a_\alpha) | \Phi' \rangle = A(\Phi^*_\alpha, \Phi'_\alpha) e^{\sum_\alpha \Phi^*_\alpha \Phi'_\alpha},$$

since $a_\alpha|\Psi'\rangle = \Psi_\alpha|\Psi'\rangle$ and $\langle\Psi|a^\dagger_\alpha = \langle\Psi|\Psi^*_\alpha$. This allows us to rewrite the exponential of the Hamiltonian as

$$\begin{aligned}
\langle \Phi_k | e^{-\epsilon\tilde{H}} | \Phi_{k+1}\rangle &= \langle \Phi_k | \Phi_{k+1}\rangle - \epsilon \langle \Phi_k | \tilde{H} | \Phi_{k+1}\rangle + \mathcal{O}(\epsilon^2) \\
&= e^{\sum_\alpha \Phi^*_{\alpha,k}\Phi_{\alpha,k+1}} (1 - \epsilon \tilde{H}(\Phi^*_{\alpha,k}, \Phi_{\alpha,k+1})) \\
&= e^{\sum_\alpha \Phi^*_{\alpha,k}\Phi_{\alpha,k+1}} e^{-\epsilon\tilde{H}(\Phi^*_{\alpha,k}, \Phi_{\alpha,k+1})} + \mathcal{O}(\epsilon^2).
\end{aligned}$$

Now, the partition function can be written as the *functional integral*

$$\begin{aligned}
Z &= \lim_{M\to\infty} \int \prod_{k=0}^{M-1} \prod_{\alpha} \frac{\mathrm{d}\Phi^*_{\alpha,k}\mathrm{d}\Phi_{\alpha,k}}{2\pi i} e^{-\sum_{k=0}^{M-1}\sum_\alpha \Phi^*_{\alpha,k}(\Phi_{\alpha,k} - \Phi_{\alpha,k+1})} \\
&\qquad \times e^{-\sum_{k=0}^{M-1} \epsilon[H(\Phi^*_{\alpha,k}, \Phi_{\alpha,k+1}) - \mu \sum_\alpha \Phi^*_{\alpha,k}\Phi_{\alpha,k+1}]} \\
&\equiv \int_{\Phi_\alpha(0) = \Phi_\alpha(\beta)} \mathcal{D}\Phi^*_\alpha(\tau) \mathcal{D}\Phi_\alpha(\tau) e^{-S[\Phi^*_\alpha(\tau), \Phi_\alpha(\tau)]},
\end{aligned}$$

with the *imaginary time* $\tau = \frac{k}{M}\beta$ and the *action*

$$S = \int_0^\beta \mathrm{d}\tau \left[\sum_\alpha \Phi^*_\alpha(\tau)(-\partial_\tau - \mu)\Phi_\alpha(\tau) + H[\Phi^*_\alpha(\tau), \Phi_\alpha(\tau)] \right]$$

Remark 3.3.

- The measures $\mathcal{D}\Phi^*_\alpha$ and $\mathcal{D}\Phi_\alpha$ should be understood as the "sum" over all complex functions $\Phi_\alpha(\tau)$ that satisfy $\Phi_\alpha(0) = \Phi_\alpha(\beta)$.
- The quantum number α labels states in the single-particle bases, e.g., momentum $\vec{q}$, position $\vec{x}$ or lattice site i.

Example 3.4 (Nonrelativistic bosons of mass m interacting via $V(\vec{x} - \vec{y})$)**.**

$$\begin{aligned}
S = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\vec{x} \Big[&\Phi^*(\vec{x},\tau) \left(-\partial_\tau - \mu - \frac{\hbar^2\nabla^2}{2m} \right) \Phi(\vec{x},\tau) \\
&+ \int \mathrm{d}^d\vec{y}\, |\Phi(\vec{x},\tau)|^2 V(\vec{x}-\vec{y}) \, |\Phi(\vec{y},\tau)|^2 \Big]
\end{aligned}$$

Example 3.5 (Higgs bosons interacting via $V(\vec{x}-\vec{y}) = \lambda\delta(\vec{x}-\vec{y})$)**.**

$$S = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\vec{x} \left[\frac{1}{2}\Phi(\vec{x},\tau)(-\partial_\mu^2 + m^2)\Phi(\vec{x},\tau) + \lambda\Phi(\vec{x},\tau)^4\right],$$

where $(\partial_\mu) = (\partial_\tau/c, \nabla)$, $\mu = 0,\ldots,d$ and m is the Higgs mass.

Example 3.6 (Dirac fermion coupled to order-parameter field)**.**

$$S = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\vec{x} \Big[\bar{\psi}(\vec{x},\tau)\gamma^\mu\partial_\mu\psi(\vec{x},\tau) + \frac{1}{2}\Phi(\vec{x},\tau)(-\partial_\mu^2 + m^2)\Phi(\vec{x},\tau) + g\Phi(\vec{x},\tau)\bar{\psi}(\vec{x},\tau)\psi(\vec{x},\tau)\Big],$$

where ψ are fermions, e.g., quarks, Φ is the order-parameter field, e.g., mesons, g is the Yukawa coupling and $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}I$ are the Dirac matrices.

Neglecting the potential, the action can be diagonalized by performing the Fourier transform

$$\Phi(\vec{r},\tau) = \frac{1}{\beta}\int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \sum_{\omega_n = \frac{2\pi}{\beta}n} \Phi(\vec{k},\omega_n)e^{i\vec{k}\vec{r}+i\omega_n\tau},$$

for $n = 0, \pm1, \pm2, \ldots$.

Example 3.7 (Nonrelativistic bosons with $V = 0$)**.**

$$S = \frac{1}{\beta}\int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \sum_{\omega_n} \left(-i\omega_n - \mu + \frac{\hbar^2k^2}{2m}\right) |\Phi(\vec{k},\omega_n)|^2,$$

yielding the propagator

$$G(\omega_n,\vec{k}) = \frac{\beta}{-i\omega_n + \frac{\hbar^2k^2}{2m} - \mu} = \beta\frac{+i\omega_n + \frac{\hbar^2k^2}{2m} - \mu}{\omega_n^2 + \left(\frac{\hbar^2k^2}{2m} - \mu\right)^2}.$$

Remark 3.8.

- For $T > 0$ all modes with $\omega_n \neq 0$, i.e., $|n| \geq 1$, have a finite gap $\Delta \propto T$.
  - Therefore they cannot contribute to a nonanalyticity in $Z = \int \mathcal{D}\Phi\mathcal{D}\Phi^* e^{-S}$ and are called *non-critical modes*.
- A system at $T > 0$ can be understood as having a finite extent β in imaginary time $\tau \in [0,\beta)$.
  - The correlation time τ_c at criticality is $\tau_\mathrm{c} \gg \beta$.

- Critical configurations have $\Phi(\tau) \approx \Phi(0)$ for all $\tau \in [0, \beta)$.
- Hence, only the $\omega_{n=0} = 0$ mode contributes to critical properties at finite T.

- Statics and dynamics decouple at $T > 0$.
- For $T = 0$ we have $\tau \in [0, \infty)$ and ω becomes continuous

$$\frac{1}{\beta} \sum_{\omega_n} \to \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}$$

  - Now, a *continuum* of small-ω modes contribute at a *quantum* critical point.
  - Quantum critical behavior in d dimensions (often) resembles classical critical behavior in $d + z$ dimensions.

3.2. Mean-Field Approximation: Landau Theory

Summary

- The mean-field approach can be obtained from field-theory in the quasistatic limit using the saddle-point approximation.

Assume a critical point at $T > 0$ and retain only the critical ($\omega_0 = 0$) modes.

Example 3.9 (Bosons with $V(\vec{x} - \vec{y}) = \lambda\delta(\vec{x} - \vec{y})$)**.** The action reads

$$S[\Phi] = \beta \int \mathrm{d}^d\vec{r} \left[\frac{\hbar^2}{2m} |\nabla\Phi(\vec{r})|^2 - \mu |\Phi(\vec{r})|^2 + \lambda |\Phi(\vec{r})|^4 \right].$$

Using $\Phi = \phi_1 + i\phi_2$ with $a = 1, 2$ real scalar fields, the action can be turned into

$$S[\vec{\phi}] = \beta \int \mathrm{d}^d\vec{r} \Big[\sum_{a=1}^{N} \frac{\hbar^2}{2m} (\nabla\phi^a)^2 - \sum_{a=1}^{N} \mu(\phi^a)^2 + \lambda \Big(\sum_{a=1}^{N} (\phi^a)^2 \Big)^2 \Big],$$

which is now as the "O(N) model" or "ϕ^4 theory".

The partition function $Z = \int \mathcal{D}\vec{\phi} e^{-S[\vec{\phi}]}$ will be dominated by configurations that *minimize* $S[\vec{\phi}]$. Hence, we can use the saddle-point approximation or mean-field approximation

$$Z \approx e^{-S[\phi_0]}, \qquad \frac{\delta S}{\delta\vec{\phi}}\bigg|_{\vec{\phi}_0} = 0, \qquad \frac{\delta^2 S}{\delta\vec{\phi}\delta\vec{\phi}^\intercal}\bigg|_{\vec{\phi}_0} \text{ is positive definite,}$$

where fluctuations $S[\vec{\phi}] - S[\vec{\phi}_0] = \frac{1}{2}\delta\vec{\phi}^\intercal \frac{\delta^2 S}{\delta\vec{\phi}\delta\vec{\phi}^\intercal}\delta\vec{\phi} + \mathcal{O}(\phi^3)$ are neglected. The free energy is

$$F = -k_B T \ln Z = k_B T S[\vec{\phi}_0],$$

such that we recover Landau-Ginzburg theory with parameters

$$(a, b, \xi_0^2) \propto (-2\mu, 4\lambda, \hbar^2/2m).$$

4. Renormalization Group

4.1. Concept of the Renormalization Group

Summary

- In Renormalization Group (RG), high-energy modes are successively integrated out and couplings become scale dependent.

In RG, one assumes that the relevant physics describing phases and phase transition is governed by the behavior at large length scales, i.e., low energies, with typical length scales $L \gg a$ much greater than the microscopic length scales a.

Example 4.1 (Magnet). In a magnet $\langle S_i^z S_{i+1}^z \rangle \neq 0$ for all T, but

$$\lim_{|i-j|\to\infty} \langle S_i^z S_j^z \rangle \begin{cases} = 0, & T > T_\mathrm{c}, \\ \neq 0, & T < T_\mathrm{c}. \end{cases}$$

The idea of RG is then to successively integrate out short-distance, i.e., high-energy, modes to obtain an *effective theory* at large length scales and low energies, such that

$$S(g_1, g_2, \ldots) \mapsto S(g_1', g_2', \ldots)$$

with action S and coupling g. Then the couplings become *scale-dependent* as $g_i \to g_i(L)$. The RG flow is the change of these couplings under the successive integration of modes, often depicted in RG flow diagrams such as in Fig. 10.

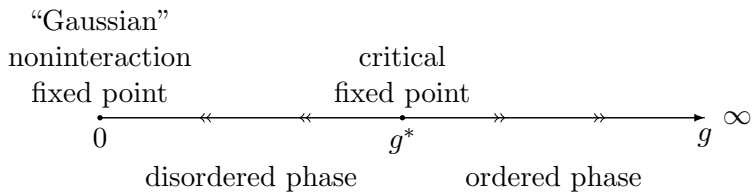


Figure 10: Exemplary RG flow diagram with critical coupling g^*.

4.2. Scaling Transformation and Scaling Dimension

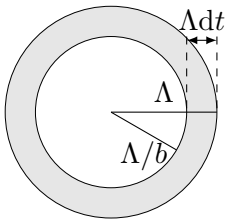
Summary

- Beta functions describe the RG flow $\dot{g}_i = \beta_i(g_j)$ in RG time $t = \int_0^t \mathrm{d}t'$ with momentum scaling $b = e^{\mathrm{d}t'}$.
- Scaling dimension $\dim[g]$ of coupling are the first order terms in the RG flow and say wether a coupling is relevant $\dim[g] > 0$, marginal $\dim[g] = 0$ or irrelevant

$\dim[g] < 0$.

- The RG flow may have fixed points, which are stable (all coupling are irrelevant), critical (exactly one coupling is irrelevant), multicritical or unstable.

We integrate out high-energy modes with momenta $\Lambda/b \leq |\vec{k}| < \Lambda$, given by the momentum shell



The infinitesimal RG step is $b = e^{\mathrm{d}t}$ with $\mathrm{d}t \ll 1$ and $t = \int_0^t \mathrm{d}t'$ the *RG time*. The RG flow is given by the *beta functions* β_i defined as

$$\frac{\mathrm{d}g_i}{\mathrm{d}t} = \beta_i(g_j),$$

with the fixed points of the flow satisfying

$$\left.\frac{\mathrm{d}g_i}{\mathrm{d}t}\right|_{g_i^*} = 0.$$

Around the Gaussian fixed point ($g^* = 0$ in a theory with one coupling g), the RG flow can be linearized as

$$\beta(g) = \theta g + \mathcal{O}(g^2),$$

with $\theta = \dim[g] = \text{const.}$ the *scaling dimension* of g. The integrated flow $\mathrm{d}g/\mathrm{d}t = \theta g$ yields $g(t) = g(0)e^{\theta t}$.

Classification of couplings. According to the scaling dimension, we classify couplings as

- a *relevant coupling* if $\dim[g] > 0$ and g increases in RG time,
- a *irrelevant coupling* if $\dim[g] < 0$ and g decreases in RG time and
- a *marginal coupling* if $\dim[g] = 0$ and higher-order terms decide its fate (marginally relevant, marginally irrelevant or exactly marginal).

Classification of fixed points. Similarly, the fixed points can be classified as

- a *stable fixed point* if all couplings are irrelevant near the fixed point,
- a *critical fixed point* if exactly one relevant direction exists,
- a *multicritical fixed point* if the number of relevant directions is $2 \leq n \leq n_0$ where n_0 is the number of tuning parameters,
- a *unstable fixed point*, if the number of relevant directions is $n > n_0$.

4.3. Momentum-Shell RG for the O(N) model

Summary

- Momentum-shell RG consists of three steps,
  1. eliminating fast modes by defining $\phi = \phi_s + \phi_f$ and integrating out ϕ_f,
  2. rescaling of momenta $k \mapsto k' = bk$, and
  3. introducing renormalized fields $\phi'(k') = b^y \phi_s(k'/b)$.
- The scaling dimensions can agree with the power-counting dimensions.
- At the Gaussian fixed point, the couplings are irrelevant above the upper critical dimension, where it governs the critical behavior.
- At the Wilson-Fisher fixed point, the couplings remain non-zero. It governs the behavior below the critical dimension.
- The stability matrix linearizes the flow near the Wilson-Fisher fixed point and gives the scaling dimensions of the couplings.

The O(N) model with tuning parameter (or mass) r and self-interaction coupling u has the action

$$\begin{aligned} S &= \int \mathrm{d}^d\vec{x} \left[\frac{1}{2}(\nabla\phi^a(\vec{x}))^2 + \frac{r}{2}(\phi^a(\vec{x}))^2 + \frac{u}{4!}(\phi^a(\vec{x})^2)^2\right] \\ &= \int_0^\Lambda \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\frac{1}{2}\phi^a(-\vec{k})(k^2+r)\phi^a(\vec{k}) \\ &\quad + \frac{u}{4!}\int_0^\Lambda \frac{\mathrm{d}^d\vec{k}_1\mathrm{d}^d\vec{k}_2\mathrm{d}^d\vec{k}_3}{(2\pi)^{3d}}\phi^a(\vec{k}_1)\phi^a(\vec{k}_2)\phi^a(\vec{k}_3)\phi^a(-\vec{k}_1-\vec{k}_2-\vec{k}_3), \end{aligned}$$

where we have rescaled $\xi_0^2(\phi^a)^2 \mapsto (\phi^a)^2$ and introduced an ultraviolet momentum cutoff Λ, $0 \leq |\vec{k}| \leq \Lambda$, with, e.g. $\Lambda \sim \frac{\pi}{a}$.

The RG transformation now consists of three stages.

1. Eliminate "fast" modes ϕ_f with momenta $\frac{\Lambda}{b} \leq |\vec{k}| \leq \Lambda$ by defining

$$\phi(\vec{k}) \equiv \underbrace{\Theta(\Lambda/b - |\vec{k}|)\phi_s(\vec{k})}_{\text{slow modes}} + \underbrace{\Theta(|\vec{k}| - \Lambda/b)\phi_f(\vec{k})}_{\text{fast modes}}.$$

2. Rescale momenta $\vec{k} \mapsto \vec{k}' = b\vec{k}$ with $0 \leq |\vec{k}'| < \Lambda$ for slow modes.
3. Introduce "renormalized" fields $\phi'(\vec{k}') = b^y \phi_s(\vec{k}'/b)$ with y chosen such that the new action in terms of ϕ' has the same coefficient for the kinetic term.

Example 4.2 (RG for the Gaussian model with $u = 0$)**.** We first perform the mode elimination, starting from partition function

$$Z = \int \mathcal{D}\phi_s \int \mathcal{D}\phi_f e^{-S_0[\phi_s, \phi_f]}$$

with

$$S_0[\phi_s, \phi_f] = \underbrace{\int_0^{\Lambda/b} \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{2} \phi_s(-\vec{k})(\vec{k}^2 + r)\phi_s(\vec{k})}_{S_{\text{eff}}[\phi_s]} + \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{2} \phi_f(-\vec{k})(\vec{k}^2 + r)\phi_f(\vec{k}).$$

The partition function may hence be written as

$$Z = \text{const.} \times \int \mathcal{D}\phi_s e^{-S_{\text{eff}}[\phi_s]}$$

with the constant $Z_{0f} = \int \mathcal{D}\phi_f e^{-S_{0f}} = [\det_{\Lambda/b \leq k < \Lambda}(k^2 + r)]^{-1/2}$.

Second, we perform the momentum rescaling in the effective action with $\vec{k}' = b\vec{k}$ and $\mathrm{d}^d\vec{k}' = b^d \mathrm{d}^d\vec{k}$, such that

$$\begin{aligned} S_{\text{eff}} &= \int_0^{\Lambda/b} \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{2} \phi_s(-\vec{k})(\vec{k}^2 + r)\phi_s(\vec{k}) \\ &= \int_0^{\Lambda} \frac{\mathrm{d}^d\vec{k}'}{(2\pi)^d} b^{-d} \frac{1}{2} \phi_s(-\vec{k}'/b)(b^{-2}\vec{k}'^2 + r)\phi_s(\vec{k}'/b). \end{aligned}$$

Third, we renormalize the fields with $\phi'(\vec{k}') = b^y \phi_s(\vec{k}'/b)$ yielding

$$S_{\text{eff}} = \int_0^{\Lambda} \frac{\mathrm{d}^d\vec{k}'}{(2\pi)^d} \frac{1}{2} \phi'(-\vec{k}')(b^{-d-2-2y}\vec{k}'^2 + b^{-d-2y} r)\phi'(\vec{k}'),$$

which has the same form as the original action for

$$y = -\frac{d+2}{2}.$$

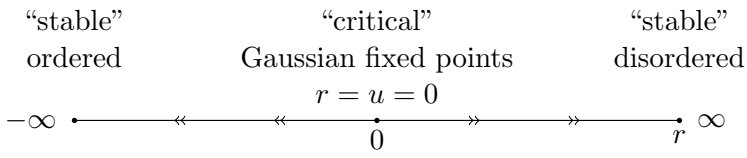
Then,

$$Z = \int \mathcal{D}\phi' e^{-\int_0^\Lambda \frac{1}{2}\phi'(\vec{k}'^2 + r')\phi'}$$

with $r' = b^2 r$. Hence, the beta function for the coupling r is

$$\beta_r = \frac{\mathrm{d}r}{\mathrm{d}\ln b} = 2r.$$

This corresponds to the follow RG flow diagram:



Example 4.3 (RG for ϕ^4 model with $u > 0$ and $N = 1$)**.** We start with the mode elimination and obtain the partition function

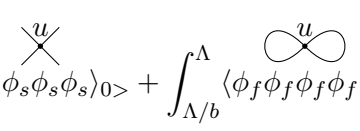
$$\begin{aligned} Z &= \int \mathcal{D}\phi_s \int \mathcal{D}\phi_f e^{-S_{0s} - S_{0f} - S_{\mathrm{int}}[\phi_s, \phi_f]} \\ &= \int \mathcal{D}\phi_s e^{-S_{0s}} \int \mathcal{D}\phi_f e^{-S_{0f}} (1 - S_{\mathrm{int}}[\phi_s, \phi_f] + \mathcal{O}(u^2)) \end{aligned}$$

with

$$S_{\mathrm{int}}[\phi_s, \phi_f] = \frac{u}{4!}\left[\int_0^{\Lambda/b} \phi_s\phi_s\phi_s\phi_s + \int_{\Lambda/b}^{\Lambda} \phi_f\phi_f\phi_f\phi_f + \binom{4}{2}\int_0^{\Lambda} \phi_s\phi_s\phi_f\phi_f\right].$$

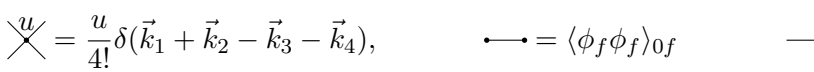
Again, let $Z_{0f} = \int \mathcal{D}\phi_f e^{-S_{0f}}$ and denote by $\langle\cdot\rangle \equiv \int \mathcal{D}\phi_f e^{-S_{0f}}(\cdot)/Z_{0f}$ the average with respect to the action of the high-energy modes. The partition function is then

$$Z = Z_{0f}\int \mathcal{D}\phi_s e^{-S_{0s}}\Bigg(1 - \frac{u}{4!}\Big[\int_0^{\Lambda/b} \langle\phi_s\phi_s\phi_s\phi_s\rangle_{0>} + \int_{\Lambda/b}^{\Lambda} \langle\phi_f\phi_f\phi_f\phi_f\rangle_{0>}$$



$$+ \binom{4}{2}\int_0^{\Lambda} \langle\phi_s\phi_s\phi_f\phi_f\rangle_{0f}\Big] + \mathcal{O}(u^2)\Bigg).$$

By Wick's theorem, we have $\langle\phi_s\phi_s\phi_f\phi_f\rangle_{0f} = \langle\phi_s\phi_s\rangle_{0f}\langle\phi_f\phi_f\rangle_{0f}$. This theory has the Feyman rules for vertex, internal and external line as follows:



Now, only *one-particle irreducible connected* (1PI) diagrams, i.e., diagrams that remain connected after cutting one internal line, contribute to the RG flow. We reexponentiate the partition function to obtain

$$Z = Z_> \int \mathcal{D}\phi_s e^{-S_{0s} - \frac{u}{4!}\left[\int_0^{\Lambda/b} \phi_s^4 + \binom{4}{2}\left(\int_0^{\Lambda/b} \phi_s^2\right)\left(\int_{\Lambda/b}^{\Lambda} \frac{1}{k^2+r}\right)\right] + \mathcal{O}(u^2)}.$$

From Leibniz rule it follows that

$$\int_{\Lambda/b}^{\Lambda} \frac{1}{k^2+r} = \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{\Lambda^2+r} \ln b + \mathcal{O}(\ln^2 b).$$

Second, we rescale momenta $\vec{k}' = b\vec{k}$ to obtain the effective action

$$S_{\text{eff}} = S_{0s} + \frac{u}{4!}\left[\int_0^{\Lambda} b^{-2d}\phi_s^4 + \binom{4}{2}\int_0^{\Lambda} b^{-d}\phi_s^2 \frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}\ln b\right] + \mathcal{O}(u^2, \ln^2 b).$$

Lastly, we renormalize the fields as $\phi'(\vec{k}') = b^{-y}\phi_s(\vec{k}'/b)$ to obtain

$$S_{\text{eff}} = \int_0^{\Lambda} \frac{1}{2}\phi'\left(\vec{k}'^2 + b^2 r + \frac{u}{2}\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}\ln b\right)\phi' + \frac{u}{4!}b^{4-d}\int_0^{\Lambda}(\phi')^4 + \mathcal{O}(u^2, \ln^2 b).$$

Then,

$$r' = b^2 r + \frac{u}{2}\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}\ln b + \mathcal{O}(u^2, \ln^2 b), \qquad u' = b^{4-d}u + \mathcal{O}(u^2, \ln^2 b).$$

By introducing the dimensionless variables $r \mapsto t = \frac{r}{\Lambda^2}$ and $u \mapsto g = \frac{S_d}{(2\pi)^d}\frac{u}{\Lambda^{4-d}}$, we obtain the beta functions

$$\beta_t = \frac{\mathrm{d}t}{\mathrm{d}\ln b} = 2t + \frac{g}{2}\frac{1}{1+t} + \mathcal{O}(g^2), \qquad \beta_g = \frac{\mathrm{d}g}{\mathrm{d}\ln b} = (4-d)g + \mathcal{O}(g^2).$$

Remark 4.4. Regarding Example 4.3, we note the following.

- The scaling dimensions $\dim[r] = 2$ and $\dim[u] = 4 - d$ agree with power-counting dimensions, which from

  $$0 = [S] = [\nabla^2] + [\phi^2] + [\mathrm{d}^d x] = 2 + 2[\phi] - d \implies [\phi] = \frac{d-2}{2}$$

  imply that $[r] = 2$ and $[u] = 4 - d$.

- To compute the leading interaction correction to β_g we need to compute the g^2 contribution. This is given diagrammatically as

  $$= (-1)\frac{1}{2!}\left(-\frac{u}{4!}\right)^2 (\phi\phi\phi\phi)(\phi\phi\phi\phi)\binom{4}{2}^2 2$$
  $$= -\frac{1}{4!}\frac{3}{2}u^2\phi^4 \int_{\Lambda/b}^{\Lambda} \frac{1}{(k^2+r)^2}$$

with (-1) from the reexponentiation, $\frac{1}{2!}$ from expanding the exponential, $\binom{4}{2}^2$ ways to choose the two contracted phi on each vertex and 2 ways to assign the contracted pairs and where $\overset{\sqcap}{\phi\phi} = \langle\phi_f\phi_f\rangle_{0f}$. Thus,

$$\beta_g = (4-d)g - \frac{3}{2}g^2\frac{1}{(1+t)^2} + \mathcal{O}(g^3).$$

- The generalization to the O(N) model yields

$$\beta_t = \frac{\mathrm{d}t}{\mathrm{d}\ln b} = 2t + \frac{N+2}{6}\frac{g}{1+t} + \mathcal{O}(g^2),$$
$$\beta_g = \frac{\mathrm{d}g}{\mathrm{d}\ln b} = (4-d)g - \frac{N+8}{6}\frac{g^2}{(1+t)^2} + \mathcal{O}(g^2).$$

The O(N) theory has two fixed points:

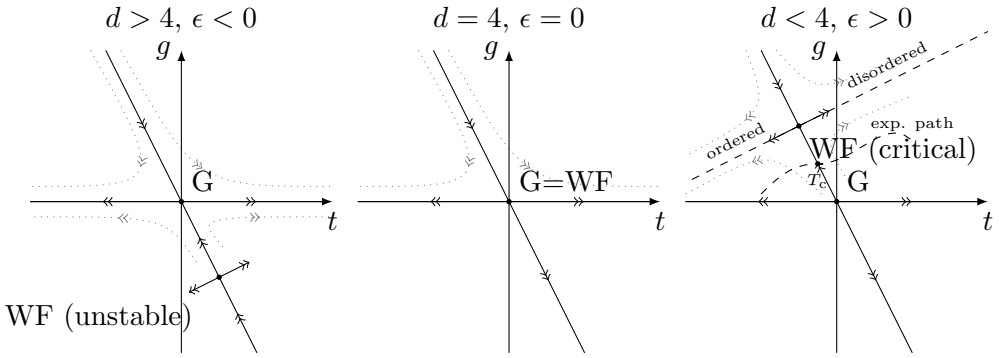
(a) The *Gaussian* fixed point at $t^* = g^* = 0$. Near this fixed point, the u (ϕ^4) and u_6 (ϕ^6) coupling have the scaling dimensions $\dim[u] = 4-d$, rendering the u coupling irrelevant for $d > 4$ and $\dim[u_6] = 6-2d$, rendering this coupling irrelevant for $d > 3$.

(b) The *Wilson-Fisher* fixed point. For *fractional dimension* $\epsilon = 4-d \ll 1$ it is at

$$g^* = \frac{6}{N+8}\epsilon + \mathcal{O}(\epsilon^2), \qquad t^* = -\frac{N+2}{2(N+8)}\epsilon + \mathcal{O}(\epsilon^2).$$

The contributions at $\mathcal{O}(\epsilon^n)$ arise from n-loop Feynman diagrams.

We obtain the following RG flow diagrams



Remark 4.5.

- The Gaussian (Wilson-Fisher) fixed point governs the critical behavior for $d > 4$ ($d < 4$).
- The upper critical dimension is $d = d_c^+ = 4$.

- For $d > d_c^+$ Landau theory becomes (asymptotically) exact because the theory is effectively Gaussian at criticality.
- An experimental system at T_c flows to the respective critical fixed point and the system becomes scale invariant.
- The critical behavior is governed by the flow in the vicinity of the critical fixed point.

Perturbations to the Wilson-Fisher fixed point. Consider the RG flow near the Wilson-Fisher fixed point with $t = t^* + \delta t$ and $g = g^* + \delta g$ with $\delta t \ll t^*$ and $\delta g \ll g^*$. Then the flow equations may be linearized, yielding for the O(N) model

$$\frac{\mathrm{d}}{\mathrm{d}\ln b}\begin{pmatrix}\delta t\\ \delta g\end{pmatrix} = \underbrace{\begin{pmatrix}2-\epsilon\frac{N+2}{N+8} & \frac{N+2}{6}\left(1+\epsilon\frac{N+2}{N+8}\right)\\ 0 & -\epsilon\end{pmatrix}}_{(B_{ij})}\begin{pmatrix}\delta t\\ \delta g\end{pmatrix} + \mathcal{O}(\delta^2)$$

defining the *stability matrix* B_{ij}. This matrix may be diagonalized $B_{ij}v_j^I = \theta^I v_i^I$.

Remark 4.6.
- The $\theta^I = \dim[v^I]$ is the *scaling dimension* of the coupling v^I at the Wilson-Fisher fixed point.
- Any *critical* fixed point has exactly one $\theta^I > 0$.

By integrating the relevant direction, we obtain $v^1(b) = v^1(0)b^{\theta^1}$ with $\theta^1 > 0$ the relevant coupling. The tuning parameter, e.g., the reduced temperature t, scales as

$$t \sim \delta t \sim v^1 \implies t \mapsto b^{\theta^1} t \implies \theta^1 = y_t.$$

Hence, the correlation-length exponent is $\nu = 1/\theta^1$.

Remark 4.7.
- At the Wilson-Fisher fixed point in the O(N) model, we have
$$\nu = \frac{1}{2} + \frac{N+2}{4(N+8)}\epsilon + \mathcal{O}(\epsilon^2),$$
while at the Gaussian $\nu = \frac{1}{2}$.
- For $N = 1$ and $\epsilon = 1$, we get
$$\nu = \frac{1}{2} + \frac{1}{12} \pm \ldots \approx 0.58$$
- Higher-order calculations for $D = 3$ yield

$\nu = 0.629(3)$	Six-loop ϵ expansion + Borel-summation
$\nu = 0.631(4)$	High-temperature expansion
$\nu = 0.6300(1)$	MC simulation
$\nu = 0.64(1)$	Neutron scattering of FeF_2

4.4. Field-theoretical perspective and anomalous dimension

Summary

- The idea of renormalized perturbation theory is to first perform RG and then perturbative expansion in the couplings.
- The anomalous dimension can be obtained from perturbation theory as

$$\langle \phi(-\vec{k})\phi(\vec{k})\rangle \propto \frac{1}{\vec{k}^2 + r - \Sigma(\vec{k})}$$

  with $r_R = r - \Sigma(0) = 0$ at criticality, hence $\Sigma(\vec{k}) - r = \Sigma(\vec{k}) - \Sigma(0)$ gives trhe anomalous dimension.

The idea is to first perform RG and then do the perturbation theory in the renormalized coupling u_R. Then, the perturbation theory does not diverge at criticality. We define

$$u_R = u - \frac{N+8}{6}u^2 \int_0^\Lambda \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\frac{1}{(k^2+r)^2} + \mathcal{O}(u^3).$$

Remark 4.8.

- The u_R is the *effective coupling* after integrating out all modes.
- The dimensionless coupling

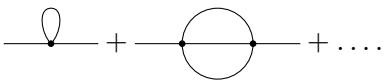
$$u \mapsto g = \frac{S_d}{(2\pi)^d}\frac{u}{|r/t|^{(4-d)/2}}$$

  diverges for $r \to 0$ when $d < d_c^+ = 4$. Hence the standard perturbation theory in u breaks down at criticality.
- The *renormalized perturbation theory* in u_R can be set up to yield finite result.

Example 4.9 (Anomalous dimension)**.** We expect the critical correlator to have the form $\langle \phi(-k)\phi(k)\rangle \propto k^{-(2-\eta)}$ with anomalous dimension η. In standard perturbation theory, this is given by

$$\langle \phi(-k)\phi(k)\rangle \propto \frac{1}{k^2 + r - \Sigma(k)}$$

with the *self-energy*

$\Sigma(k) =$  $+ \ldots.$

The tadpole contributes only to $\Sigma(0)$, because of momentum conservation and hence the sunset diagram gives the leading nontrivial momentum dependence.

Right at the critical point we have $r_R = r - \Sigma(0) = 0$. In $d = 4 - \epsilon$ dimensions, the sunset diagram yields

$$\Sigma(k) - \Sigma(0) = u^2 \left[c_1 k^2 \ln \frac{\Lambda}{k} + \mathcal{O}(k^4, \epsilon) \right] + \dots .$$

As to the leading order $u_R = u + \mathcal{O}(u^2)$, we have

$$\langle \phi(-k)\phi(k) \rangle \propto \frac{1}{k^2[1 + c_2 g_R^2 \ln \frac{\Lambda}{k}]} + \mathcal{O}(g_R^3) = \frac{1}{k^2} \left(\frac{\Lambda}{k} \right)^{-c_2 g_R^2} + \mathcal{O}(g_R^3)$$

with $g_R = g^*$ at the critical point. After reinstating the constants, we read off

$$\eta = c_2 (g^*)^2 = \frac{N+2}{2(N+8)^2} \epsilon^2 + \mathcal{O}(\epsilon^3).$$

Remark 4.10.

- The last step $1 + c_2 g_R^2 \ln \frac{\Lambda}{k} = \left(\frac{\Lambda}{k}\right)^{c_2 g^2} + \mathcal{O}(g_R^4)$ effectively resums an infinite number of diagrams.
- For $N = 1$ and $\epsilon = 1$ (the $3d$ Ising model),

  $$\eta = \frac{1}{54} + \dots \approx 0.02$$

  to be compared with the (almost exact) value from Monte-Carlo

  $$\eta_{\mathrm{MC}} = 0.0363(1).$$

4.5. Phase Transitions and Critical Dimensions

Summary

- Above the upper critical dimension mean-field theory becomes asymptotically exact.
- Below the lower critical dimension fluctuations destroy order.

Universality implies that different microscopic models flow to the same RG fixed point at criticality. The important critical dimensions are

- the *upper* critical dimension d_c^+, where mean-field theory becomes asymptotically exact for $d \geq d_c^+$,
- the *lower* critical dimension d_c^-, where fluctuations destroy the ordered phase at any temperature for $d \leq d_c^-$.

The critical exponents typically depend on d for $d_c^- < d < d_c^+$ and become d-independent for $d > d_c^+$, expect in systems with sufficiently long-ranged interactions.

Example 4.11 (Classical magnets with short-range interactions, O(N) models)**.** Here, $d_c^+ = 4$ and

$$d_c^- = \begin{cases} 2, & N > 2, \\ 1, & N = 1. \end{cases}$$

The case $N = 2$ and $d = 2$ is special.

Remark 4.12. The following physics hold near the upper critical dimension (in O(N) models):

- For $d < d_c^+ = 4$, the critical fixed point is the *Wilson-Fisher fixed point*, observables are computable in *renormalized* perturbation theory in $u^* = u^*(d)$ and hyperscaling is *valid*.
- For $d > d_c^+ = 4$, the critical fixed point is the *Gaussian fixed point*, observables are computable in *standard* perturbation theory in u, exponents take mean-field values and the hyperscaling relations are violated, which can be traced back to the presence of a *dangerously irrelevant* coupling u[1], i.e., the free energy is non-analytic at $u = u^* = 0$.
- For $d = d_c^+ = 4$, there are logarithmic correlations to the mean-field behavior.

Analytic alternatives to $\epsilon = 4 - d$ expansion are

- the $1/N$ expansion,
- the $2 + \epsilon$ expansion in $T_c(\epsilon) \propto \mathcal{O}(\epsilon)$,
- the conformal bootstrap, i.e., the use of symmetry and unitarity arguments to constrain the scaling dimension of operators assuming conformal invariance.

5. Theoretical Models for Quantum Phase Transitions

5.1. Quantum Ising Model

Summary

- The quantum Ising model is given by the Hamiltonian

$$H_I = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - Jg \sum_i \sigma_i^x$$

- For $g \ll 1$ the spins prefer to align with the z direction, while for $g \gg 1$ the spins prefer an orientation along the x direction, hence yielding a quantum critical transition.

[1] If, e.g., the order parameter still depends on the irrelevant coupling, it is called dangerously irrelevant.

The quantum Ising model is given by the Hamiltonian

$$H_{\mathrm{I}} = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - Jg \sum_i \sigma_i^x$$

defined on a regular d-dimensional lattice. Here, H_{I} describes quantum spins $\frac{1}{2}$, $\vec{S}_i = \frac{1}{2}\vec{\sigma}_i$ with $[S_i^x, S_j^y] = iS^z$, in an external magnetic field $\vec{H} \parallel \vec{e}_x$.

It exhibits the $\mathbb{Z}_2$ symmetry

$$\sigma_i^z \mapsto -\sigma_i^z.$$

The following limiting cases apply:

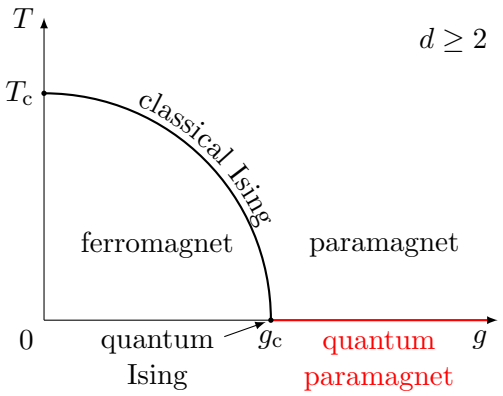
(a) For no external field $g \to 0$:

- The groundstate $|0\rangle$ is a ferromagnet with $|\Uparrow\rangle = \prod_i |\uparrow\rangle_i$ or $|\Downarrow\rangle = \prod_i |\downarrow\rangle_i$.
- The $\mathbb{Z}_2$ symmetry is spontaneously broken.
- There is long-range order with order parameter $\langle 0|\sigma_i^z|0\rangle = \pm N_0$.
- A small finite g will reduce N_0, but the system remains ordered.

(b) For vanishing coupling $g \to \infty$:

- The groundstate $|0\rangle$ is the polarized state $\prod_i |\rightarrow\rangle_i$, where $\sigma_i^x|\rightarrow\rangle_i = +|\rightarrow\rangle_i$.
- There is no spontaneous symmetry breaking.
- A large finite g will allows admixture of $|\leftarrow\rangle_i$ spins, but the system remains disordered.

Therefore, there must be a quantum phase transition at some critical $g = g_{\mathrm{c}}$. The phase diagram is



Experimental examples include $LiHoF_4$ (3d Ising with long-range dipolar interaction) and $CoNb_2O_6$ (1d Ising).

5.2. Quantum Rotor Models

Summary

- The quantum rotor model is given by the Hamiltonian

$$H_{\mathrm{R}} = -J \sum_{\langle ij \rangle} \vec{n}_i \cdot \vec{n}_j + \frac{Jg}{2} \sum_i \mathbb{L}_i^2.$$

- For $g \ll 1$ the rotors prefer to align, while for $g \gg 1$ the rotors prefer a maximally uncertain orientation, hence yielding a quantum critical transition.

A *quantum rotor* is a particle on the unit sphere in $N \geq 2$ dimensions, with orientation $\vec{n}$ and constraint $\vec{n}^2 = 1$. With the conjugate momentum $\vec{p}$, the components satisfy the commutation relation

$$[n_\alpha, p_\beta] = i\delta_{\alpha\beta},$$

for $\alpha, \beta = 1, \ldots, N$. In analogy to the classical angular momentum, the rotor angular momentum is

$$L_{\alpha\beta} \equiv n_\alpha p_\beta - n_\beta p_\alpha$$

with $N(N-1)/2$ independent components. The $L_{\alpha\beta}$ are the *generators* of the $\mathrm{O}(N)$ rotations of $\vec{n}$ on the unit sphere.

Example 5.1 (Quantum rotors for $N = 3$)**.** There are three independent components $L_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma} L_{\beta\gamma}$ with

$$[L_\alpha, L_\beta] = i\epsilon_{\alpha\beta\gamma} L_\gamma, \qquad [L_\alpha, n_\beta] = i\epsilon_{\alpha\beta\gamma} n_\gamma.$$

The kinetic energy is

$$H_{\mathrm{kin}} = \frac{Jg}{2} L_{\alpha\beta}^2 \equiv \frac{1}{2I} \mathbb{L}^2 \propto \begin{cases} L_z^2, & N = 2, \\ \vec{L}^2, & N = 3, \end{cases}$$

with moment of inertia $I = \frac{1}{Jg}$ and eigenvalues

$$\epsilon_{\mathrm{kin}} = \begin{cases} \frac{Jg}{2} l^2, & N = 2 \text{ (two-fold degenerate for } l \neq 0), \\ \frac{Jg}{2} l(l+1), & N = 3 \text{ } (2l+1 \text{ degenerate}), \end{cases}$$

with $l = 0, 1, 2, \ldots$.

On a lattice of rotors, the Hamiltonian is

$$H_{\mathrm{R}} = -J \sum_{\langle ij \rangle} \vec{n}_i \cdot \vec{n}_j + \frac{Jg}{2} \sum_i \mathbb{L}_i^2.$$

It has O(N) symmetry with $\vec{n}_i \mapsto R\vec{n}_i$ where R is a rotation matrix.

The following limiting cases apply:

(a) For high moment of inertia $g \ll 1$:

- The O(N) symmetry is spontaneously broken.
- There is long-range order $|\langle 0|\vec{n}_i|0\rangle| = N_0$ and $\lim_{|\vec{r}_i - \vec{r}_j|} \langle 0|\vec{n}_i \cdot \vec{n}_j|0\rangle = N_0^2$.

(b) For low moment of inertia $g \gg 1$:

- The ground state is symmetric.
- There is no long-range order and $\langle 0|\vec{n}_i \cdot \vec{n}_j|0\rangle \propto e^{-|\vec{r}_i - \vec{r}_j|/\xi}$.

Therefore, there is a quantum phase transition at finite $g = g_c$. The universality class is the O(N) model in $d+1$ dimensions.

Example 5.2. For $N = 2$ this is a lattice of superconducting islands and for $N = 3$ a lattice of spin pairs, called "coupled dimers".

5.3. Coupled-dimer model

The Hamiltonian is

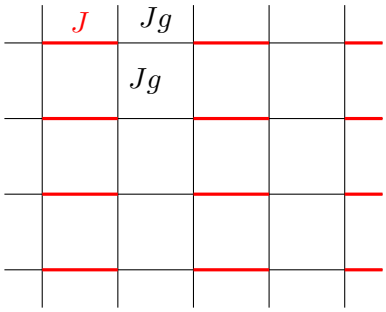
$$H_{\mathrm{CD}} = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j,$$

with

$$J_{ij} = \begin{cases} J, & \text{on intradimer bonds,} \\ \lambda J, & \text{on interdimer bonds,} \end{cases}$$

and $J > 0$ antiferromagnetic.

Example 5.3 (Coupled-dimer in 2d)**.** The following Hamiltonian is a coupled dimer in two dimensions.



The following limiting cases apply:

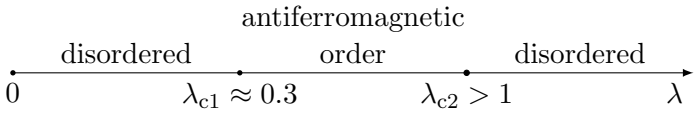
(a) For no coupling $\lambda = 0$, the dimers are spin singlets with $S = 0$, and the triplets with $S = 1$ are gapped with $\Delta = J$.

(b) For weakly-coupled dimers $\lambda \ll 1$, the system is disordered.

(c) For strongly-coupled dimers $\lambda \sim 1$, there is antiferromagnetic long-range order and the SU(2) symmetry is spontaneously broken.

(d) For decoupled spin lagges $\lambda \ll 1$, the system is disordered.

The phase diagram is



The universality class is the O(3) Heisenberg model in $d + 1$ dimensions.

Experimental examples include $TlCuCl_3$, a 3d coupled-dimer system with $\lambda \propto p$, and $BaCuSi_2O_6$, as 2d layers of dimers.

6. General Aspects of Quantum Phase Transitions

6.1. Classical and Quantum Fluctuations

Summary

- In classical systems, dynamics and statics decouple as $[T, V] = 0$, while they do not in quantum systems with $[T, V] \neq 0$.
- At finite temperature the energy scales $k_B T$ and $\hbar\omega_c \simeq \hbar/\tau_c$ compete. Near criticality, $\tau_c \to \infty$, hence any critical point is asymptotically classical.

The partition function in the canonical ensemble is

$$Z = \mathrm{tr}[e^{-\beta H}].$$

Classical systems. In classical systems, statics and dynamics decouple, at $H = T + V$ with $[T, V] = 0$. Hence,

$$Z = \mathrm{tr}[e^{-\beta T} e^{-\beta V}].$$

Example 6.1. For the kinetic energy $T = \sum_i \frac{p_i^2}{2m}$ and potential $V = V(x_i)$, the partition function factorizes as

$$Z = \underbrace{\mathrm{tr}[e^{-\beta T}]}_{\text{analytic}} \mathrm{tr}[e^{-\beta V}]$$

and any nonanalyticity can only arise from $\mathrm{tr}[e^{-\beta V}]$.

Hence, in classical systems the critical exponents (except for z) are determined by the static (momentum independent) piece of H and therefore yield *static universality classes*. The dynamical exponent z is an independent exponent.

Quantum systems. In quantum systems, statics and dynamics are generally coupled as $[T, V] \neq 0$. The dynamical exponent z is therefore an integral part of the universality class.

Quantum versus classical transitions. At criticality, both correlation length and correlation time diverge as $\xi \to \infty$ and $\tau_c \to \infty$. The energy scale for order-parameter fluctuations hence vanishes as

$$\hbar\omega_c \simeq \frac{\hbar}{\tau_c} \to 0.$$

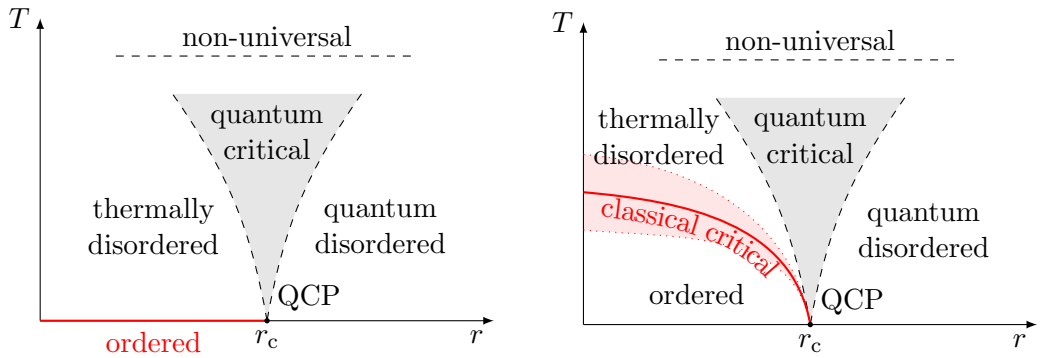
For a classical transition it has to hold that the thermal fluctuations are greater than the quantum ones, i.e., $k_B T_c \gg \hbar\omega_c$.

We conclude the following:

- Any critical point with $T_c > 0$ is asymptotically *classical*, in the sense that long-distance fluctuations are governed by classical statistical mechanics.[2]
- Any critical point at $T = 0$ is *quantum*, in the sense that quantum statistics is needed to describe order-parameter fluctuations. Often the quantum critical point is an endpoint of a line of thermal critical points.

6.2. Phenomenology: Phase Diagrams and Crossovers

[2]Note that this is not a statement about the physics underlying the different phases, e.g., in superconductivity.



(a) System without a thermal transition

(b) System with a thermal transition

Figure 11: Quantum phase diagrams for systems with and without a thermal transition. Examples of systems without thermal transition are the 1d Ising and 2d Heisenberg model. Systems with a thermal transition are the 2d Ising and 3d Heisenberg models.

Summary

- In the quanutm disordered and ordered regime there are well defined quasiparticles.
- The classical critical regime is fully described by classical power laws.
- In the quantum critical regime, no quasiparticle description is possible, with the crossover scale given by $k_B T \simeq \Delta \propto |r - r_c|^{\nu z}$.
- In the non-universal regime microscopic details become important.

Figure 11 shows the phase diagrams of quantum critical systems. There are the following regimes:

(a) Quantum disordered ($r > r_c$, T small)

- There are well-defined quasiparticles with gap $\Delta > 0$.
- The density of excited quasiparticles is $n \propto \exp(-\frac{\Delta}{k_B T})$.

(b) Ordered ($r < r_c$, $T < T_c(r)$)

- There are well-defined quasiparticles.
- The gap vanishes, i.e., $\Delta = 0$, if a continuous symmetry is spontaneously broken (Goldstone modes), but is finite of only a discrete symmetry is broken.

(c) Classical critical ($r < r_c$, $T \approx T_c(r)$)

- Classical critical power laws hold and the system is fully described within classical statistics.

- The width vanishes for $r \nearrow r_c$ ($T \searrow 0$).

(d) Thermally disordered ($r < r_c$, $T > T_c(r)$)

- Thermal fluctuations destroy long-range order.

(e) Quantum critical ($r \approx r_c$, T small)

- No quasiparticle description is possible.
- There is a gapless continuum of excitations.
- The system exhibits unconventional thermodynamic and transport properties.
- The crossover scale is $k_B T_{QC} \propto \Delta \propto \frac{\hbar}{\tau_c} \propto \left(\frac{1}{\xi}\right)^z \propto |r - r_c|^{\nu z}$.

(f) Non-universal (T large)

- Microscopic details become important.
- The crossover scale if $k_B T_{mic} \sim J$, where J is the relevant microscopic energy scale.

6.3. Quantum ϕ^4 Theory

Summary

- In quantum ϕ^4 theory, the action is given by

$$S = \int \mathrm{d}^d x \int_0^\beta \mathrm{d}\tau \left[\frac{1}{2}(\partial_\tau \phi)^2 + \frac{c^2}{2}(\partial_i \phi)^2 + \frac{r}{2}\phi^2 + \frac{u^4}{4!}\phi^4\right].$$

- Due to space-time symmetry, $z = 1$ and the quantum critical point is a classical critical point in $d + z$ dimensions.

The partition function of the quantum ϕ^4 theory is

$$Z = \int \mathcal{D}\phi(\vec{x}, \tau) e^{-S[\phi]}$$

with action

$$S = \int \mathrm{d}^d x \int_0^\beta \mathrm{d}\tau \left[\frac{1}{2}(\partial_\tau \phi)^2 + \frac{c^2}{2}(\partial_i \phi)^2 + \frac{r}{2}\phi^2 + \frac{u^4}{4!}\phi^4\right]$$

with $(\partial_i)_{i=1,\dots,d} = \nabla$ and c some velocity.

Remark 6.2.

- The form of S follows from $\mathbb{Z}_2$ symmetry $\phi \mapsto -\phi$ and time-reversal invariance $\tau \mapsto -\tau$.

- Higher-order terms $\propto \phi^6$ or $\propto \partial^2\phi^4$, etc., are irrelevant in $d > 2$.
- The dynamial exponent $z = 1$ from $\tau_\mathrm{c} \propto \xi^z$ due to space-time symmetry.
- For $T = 0$ ($\beta \to \infty$) τ enters as an additional spatial dimension.
- From the quantum-to-classical mapping, the quantum critical point in d dimensions is the classical critical point in $d + z$ dimensions.
- In general, z may take any value.
- For $T > 0$ ($\beta < \infty$) fluctuations for $\tau_\mathrm{c} \gtrsim \beta$ are frozen, implying $\phi(\vec{x}, \tau) \simeq \phi(\vec{x})$ and $\int_0^\beta \mathrm{d}\tau \simeq \beta$, i.e., the theory becomes classical.
- In the quantum critical regime $\tau_\mathrm{c} \sim \beta$ and therefore thermal and quantum fluctuations are equally important.

6.4. Quantum Scaling Hypothesis

Summary

- The quantum scaling hypothesis includes the absolute temperature in the singular part of the free energy as

$$f_s(t, h, T) = b^{-d-z} f_s(b^{y_t} t, b^{y_h} h, b^{y_T} T).$$

- The specific heat and Grüneisen parameter, i.e., the ratio between thermal expansion $\partial S/\partial T \propto (\partial V/\partial T)_p$ and specific heat, are signatures of the quantum critical regime.

The classical scaling hypothesis of the singular part of the free energy density was $f_s(t, b) = b^{-d} f_s(tb^{y_t}, hb^{y_h})$ for $d < d_\mathrm{c}^+$ with tuning parameter t, e.g., the reduced temperature, and external source h, e.g., the magnetic field, and rescaling factor $b > 1$.

The quantum scaling hypothesis now includes the absolute temperature as

$$f_s(t, h, T) = b^{-(d+z)} f_s(tb^{y_t}, hb^{y_h}, Tb^{y_T}),$$

where the tuning parameter t, i.e., the *distance to criticality*, is, e.g., $t = \frac{r - r_\mathrm{c}}{r_\mathrm{c}}$ in quantum ϕ^4 theory, or $t = \frac{p - p_\mathrm{c}}{p_\mathrm{c}}$ for a pressure-induced quantum critical point, etc. The scaling transformations are now

$$x \mapsto bx, \qquad\qquad \tau \mapsto b^z \tau.$$

The correlation-length exponent is still $\nu = 1/y_t$, while the dynamical critical exponent follows from $T \propto \tau^{-1}$ as

$$\frac{1}{\tau} \mapsto b^{-z} \frac{1}{\tau'} \propto b^{-z} T' = b^{-z} b^{y_T} T \implies z = y_T.$$

By setting $bt^{y_t} = bt^{1/\nu} = 1$, we obtain the scaling relation

$$f_s(t,h,T) = |t|^{\nu(d+z)} f_s\left(\pm 1, \frac{h}{|t|^{\nu y_h}}, \frac{T}{|T|^{\nu y_T}}\right) \equiv |t|^{\nu(d+z)} F_\pm\left(\frac{h}{|t|^{\nu y_h}}, \frac{T}{|T|^{\nu y_T}}\right)$$

with scaling function $F_\pm$.

From the second component, we find the finite-temperature behavior

$$\frac{T}{|t|^{\nu z}} \begin{cases} \gg 1, & \text{quantum critical regime,} \\ \ll 1, & \text{stable phase regime.} \end{cases}$$

Hence, the crossover line are at $T \sim |t|^{\nu z}$.

Example 6.3 (Specific heat in the quantum critical regime)**.** For no external field, i.e., $h = 0$, we have the scaling relation

$$f_s(t,T) = |t|^{\nu(d+z)} F_\pm(0, T/|t|^{\nu z}) = T^{\frac{d+z}{z}} \tilde{F}_\pm(T/|t|^{\nu z}),$$

with $\tilde{F}_\pm \equiv x^{-\frac{d+z}{z}} F_\pm(0,x)$. In the quantum critical regime $T/|t|^{\nu z} \to 0$ and hence

$$f_s(0,T) = T^{\frac{d+z}{z}} \times \text{const.},$$

leading to the quantum critical scaling of the specific heat as

$$C_{t=0} = VT\frac{\partial^2 f_s}{\partial T^2} \propto T^{\frac{d}{z}}$$

for $d < d_c^+$.

Grüneisen parameter. To probe for the quantum critical point, one can use the Grüneisen parameter Γ defined as

$$\Gamma \equiv \frac{B}{C}$$

with $B = \partial S/\partial t$.

Example 6.4 (Grüneisen parameter for a pressure-induced QCP)**.** Let the tuning parameter be $t = (p - p_c)/p_c$ and

$$B = \frac{\partial S}{\partial t} \propto \left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p \propto \alpha$$

with α the *thermal expansion coefficient*. In the last step, a Maxwell relation was used. Hence,

$$\Gamma \propto \frac{\alpha}{C_p}.$$

The Grüneisen parameter scales differently at a thermal critical point than at a quantum critical point. Using $S = \partial f_s / \partial T$, we find for the thermal expansion

$$B \propto \frac{\partial^2 f_s}{\partial t \partial T} = |t|^{\nu d - 1} F_{\pm}^{(B)}(T/|t|^{\nu z}) = T^{\frac{d}{z} - \frac{1}{\nu z}} \tilde{F}_{\pm}^{(B)}(T/|t|^{\nu z}).$$

Similarly, for the specific heat

$$C \propto T \frac{\partial^2 f_s}{\partial T^2} = |t|^{\nu d} F_{\pm}^{(C)}(T/|t|^{\nu z}) = T^{\frac{d}{z}} \tilde{F}_{\pm}^{(C)}(T/|t|^{\nu z}).$$

Hence, the Grüneisen parameter is

$$\Gamma = \frac{B}{C} = \begin{cases} G_t |t|^{-1}, & T = 0, t \to 0, \\ G_T T^{-\frac{1}{\nu z}}, & t = 0, T \to 0, \end{cases}$$

and Γ diverges at a quantum critical point, but not at a thermal critical point with only one tuning parameter t. This is a unique signature of quantum criticality.

6.5. Quantum-to-Classical Mapping

Summary

- The quantum-to-classical correspondence follows from writing the partition function with a classical Hamiltonian H_c or equivalently with a quantum Hamiltonian H_q as

$$Z = \sum_i e^{-H_c(i)} = \operatorname{tr} e^{-H_q}.$$

6.5.1. Classical Ising Chain

Summary

- The transfer-matrix formalism allows to solve the quantum Ising model.
- The quantum-to-classical correspondence follows from

$$Z = \operatorname{tr} T^M \simeq \operatorname{tr} e^{-H_q/T}$$

for appropriate Hamiltonian H_q with "temperature" $T \propto M^{-1} \propto L^{-1}$ and gap $\Delta \propto \xi^{-1}$.

The Hamiltonian is

$$H = -K \sum_{i=1}^{M} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{M} \sigma_i$$

with $\sigma_i = \pm 1$, periodic boundary conditions $\sigma_{M+1} = \sigma_1$, and the dimensionless parameters $K = \frac{J}{k_B T}$ and $h = \frac{H_0}{k_B T}$.

The *transfer matrix method* allows to compute the partition function as a matrix product

$$Z = \sum_{\{\sigma_i\}} e^{-H} = \sum_{\{\sigma_i\}} \prod_{i=1}^{M} T(\sigma_i, \sigma_{i+1})$$

with $T(\sigma_i, \sigma_j) = \exp(K\sigma_i\sigma_j + \frac{h}{2}(\sigma_i + \sigma_j))$. In matrix notation, this yields the *transfer matrix* T as

$$T(\sigma_i, \sigma_j) = \langle \sigma_i | T | \sigma_j \rangle \implies T = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}.$$

The partition sum can then be evaluated as

$$Z = \sum_{\{\sigma_i\}} \prod_{i=1}^{M} \langle \sigma_i | T | \sigma_{i+1} \rangle = \sum_{\{\sigma_i\}} \langle \sigma_1 | T | \sigma_2 \rangle \langle \sigma_2 | T | \sigma_3 \rangle \dots \langle \sigma_M | T | \sigma_1 \rangle = \operatorname{tr} T^M.$$

By diagonalizing T as $T = ODO^\intercal$ with $O^\intercal O = I$ and $D = \operatorname{diag}(\epsilon_1, \epsilon_2)$ with eigenvalues

$$\epsilon_{1,2} = e^K \cosh h \pm \sqrt{e^{2K} \sinh^2 h + e^{-2K}},$$

the partition sum can be computed as

$$Z = \epsilon_1^M + \epsilon_2^M.$$

Example 6.5 (Correlation function at $h = 0$)**.** The transfer matrix framework allows to compute the correlation function as

$$\langle \sigma_i \sigma_j \rangle = \frac{1}{Z} \sum_{\{\sigma_i\}} \sigma_i \sigma_j e^{-H} = \frac{1}{Z} \operatorname{tr}[T^{i-1} S T^{j-i} S T^{M-j+1}] = \frac{\epsilon_1^{M-j+i} \epsilon_2^{j-i} + \epsilon_2^{M-j+i} \epsilon_1^{j-i}}{\epsilon_1^M + \epsilon_2^M},$$

having defined $S = \operatorname{diag}(1, -1)$. In the thermodynamic limit ($M \to \infty$), this turns into

$$\langle \sigma_i \sigma_j \rangle \to \left(\frac{\epsilon_2}{\epsilon_1} \right)^{j-i} = (\tanh K)^{j-i}.$$

Now, defining $\tau = ja$ and $\sigma(\tau) = \sigma_j$, we find the scaling limit ($K \to \infty$) as

$$\langle \sigma(\tau) \sigma(0) \rangle = e^{-\frac{|\tau|}{\xi}}$$

with the correlation length $\xi = a / \ln \coth K \simeq \frac{a}{2} e^{2K} \gg 1$.

6.5.2. Scaling Limit and Universality

Summary

- Taking the scaling limit $\xi/a \to \infty$ and $h/a \to \infty$ with allows to derive the scaling functions for the Ising model in terms of L/ξ and hL.

The relevant length scales in our problem are

- large, as the correlation length ξ, the observation scale τ, the systemsize $L_\tau = Ma$, or
- small, as the lattice constant a.

In the scaling limit, we assume that large length scales are much larger than small length scales, i.e., go to the continuum limit.

Example 6.6. We send $a \to 0$, while ξ, τ and L_τ are kept finite.

Example 6.7 (Universality of the free-energy density)**.** For the classical Ising model, the free-energy density is

$$f = -\frac{\ln Z}{Ma} = -\frac{\ln(\epsilon_1^M + \epsilon_2^M)}{Ma},$$

with transfer-matrix eigenvalues

$$\epsilon_{1,2} \simeq \sqrt{\frac{2\xi}{a}}\left(1 \pm \frac{a}{2\xi}\sqrt{1 + 4\tilde{h}^2\xi^2}\right)$$

for $2\xi/a \simeq e^{2K} \gg 1$ and $\tilde{h} = h/a \ll 1$. Now, for $a \to 0$, this yields the free-energy density

$$f = \epsilon_0 - \frac{1}{L_\tau}\underbrace{\ln\left[2\cosh\left(L_\tau\sqrt{(2\xi)^{-2} + \tilde{h}^2}\right)\right]}_{\Phi_f(L_\tau/\xi, \tilde{h}L_\tau)}$$

with $\epsilon_0 = -K/a$ the ground-state energy for $T \to 0$ and $\tilde{h} = 0$. The *universal* scaling function

$$\Phi_f(x, y) = -\ln\left[2\cosh\left(\sqrt{\left(\frac{x}{2}\right)^2 + y^2}\right)\right]$$

is valid for all 1d systems with $\mathbb{Z}_2$ symmetry in the scaling limit!

Example 6.8 (Universality of the correlation function)**.** For $\tilde{h} = 0$, the correlation function is

$$\langle\sigma(\tau)\sigma(0)\rangle = \frac{e^{-|\tau|/\xi} + e^{-(L_\tau - |\tau|)/\xi}}{1 + e^{-L_\tau/\xi}} \xrightarrow{L_\tau \gg \xi} e^{-|\tau|/\xi},$$

as before, where in the last step we took the thermodynamic limit. The scaling function is then

$$\langle\sigma(\tau)\sigma(0)\rangle = \Phi_\sigma(\tau/L_\tau, L_\tau/\xi, \tilde{h}L_\tau).$$

6.5.3. Mapping to a Quantum Spin

Summary

- The Ising model maps to one quantum spin.

In the scaling limit, the classical Ising Hamiltonian can be mapped to a quantum spin as follows from the transfer matrix

$$T = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix} = e^{K+h}\left(\frac{I+\sigma^z}{2}\right) + e^{K-h}\left(\frac{I-\sigma^z}{2}\right) + e^{-K}\sigma^x$$
$$= e^{K}(I_2 + h\sigma^z + e^{-2K}\sigma^x) + \mathcal{O}(h^2).$$

By using $K = -\epsilon_0 a$, $h = a\tilde{h}$ and $e^{-2K} \simeq a/2\xi$ for $a \to 0$, we find

$$T = e^{-a(\epsilon_0 - \frac{1}{2\xi}\sigma^x - \tilde{h}\sigma^z)} + \mathcal{O}(h^2, a^2) \equiv e^{-aH_{\mathrm{qu}}}$$

with *Hamiltonian operator*

$$H_{\mathrm{qu}} = \epsilon_0 - \frac{\Delta}{2}\sigma^x - \tilde{h}\sigma^z$$

and $\Delta = \xi^{-1}$.

The partition function of the classical system changes as

$$Z = \operatorname{tr} T^M = \operatorname{tr} e^{-H_{\mathrm{qu}}/T}$$

with "temperature" $T = 1/L_\tau = (Ma)^{-1}$. Now, Z represents the partition function of a single quantum spin 1/2 in two perpendicular field $\tilde{h}$ and $\Delta/2$ at a "quantum temperature" $T = 1/L_\tau$.

Similarly, the free energy

$$F = E_0 - T \ln\left[2\cosh\left(\frac{1}{T}\sqrt{\frac{\Delta^2}{4} + \tilde{h}^2}\right)\right]$$

is equivalent to the energy of a quantum spin 1/2 in external field $\left|\vec{H}_0\right| = \sqrt{\Delta^2/4 + \tilde{h}^2}$.

In conclusion, a quantum system at temperature T is equivalent to a classical system of finite length $L_\tau = 1/T$. This can is summarized in the following quantum-to-classical correspondence

one-dimensional classical spin chain	zero-dimensional quantum spin
system size L_τ	inverse temperature $1/T$
correlation length ξ	inverse excitation gap $1/\Delta$

Remark 6.9. In classical systems the temperature can always be absorbed in rescaling of the coupling, while in quantum systems temperature enters as an independent parameter.

6.5.4. Mapping the XY Chain to the O(2) Quantum Rotor

Summary

- The classical XY chain maps to an O(2) quantum rotor.

The classical XY chain is given by

$$H_{\text{cl}} = -K \sum_{i=1}^{M} \vec{n}_i \cdot \vec{n}_{i+1} - \sum_{i=1}^{M} \vec{h} \cdot \vec{n}_i$$

with $\vec{n}_i = (n_i^x, n_i^y) \in S^1$, i.e., $\vec{n}_i^2 = 1$ for all $i = 1, \ldots, M$.

The equivalent quantum system with $\vec{h} \parallel \vec{e}_x$ is

$$H_{\text{qu}} = -\Delta \frac{\partial^2}{\partial \theta^2} - \tilde{h} \cos\theta$$

where $\vec{n}_i \mapsto \vec{n}(\tau) = (\cos\theta(\tau), \sin\theta(\tau))$ with $\tau = ia$, $\tilde{h} = h/a$ and $\Delta = \xi^{-1}$ the *energy gap* as before.

6.5.5. Mapping the Heisenberg Chain to the O(3) Quantum Rotor

Summary

- The Heisenberg chain maps to an O(3) quantum rotor.

The classical Heisenberg chain is given by

$$H_{\text{cl}} = -K \sum_{i=1}^{M} \vec{n}_i \cdot \vec{n}_{i+1} - \sum_{i=1}^{M} \vec{h} \cdot \vec{n}_i$$

with *classical spins* $\vec{n}_i = (n_i^x, n_i^y, n_i^z) \in S^2$.

The equivalent quantum system is

$$H_{\text{qu}} = -\frac{\Delta}{2} \vec{L}^2 - \tilde{\vec{h}} \cdot \vec{n}$$

with angular momentum operator $\vec{L}$.

Remark 6.10.

- The classical spin maps to O(3) quantum rotor, not to a quantum spin.
- Quantum spins have nontrivial dynamics, so-called "Berry-phase terms", without classical analogs.

6.5.6. Rules and Exceptions for the Quantum-to-Classical Correspondence

Summary

- A QCP in d dimensions is equivalent to a TCP in $d+z$ dimensions.

As a general rule

a QCP in d dimensions is equivalent to a TCP in $d+z$ dimensions.

Remark 6.11.

- The correspondence applies in the scaling limit, i.e., for a QCP at $T \to 0$.
- The real-time dynamics at $T = 0$ may be obtained by *Wick rotation* $\tau \mapsto it$ such that

$$Z = \int \mathcal{D}\phi e^{iS[\phi]} \leftrightarrow \int \mathcal{D}\phi e^{-S[\phi]}.$$

  The analytic continuation is typically not easily obtained.

Remark 6.12. The following exceptions to the quantum-to-classical correspondence need to be considered.

- Systems with quenched disorder, i.e., disorder frozen in imaginary time.
- Systems with quantum of Berry-phase dynamics, e.g., single quantum particle

$$S = \int \mathrm{d}\tau \left[\phi \partial_\tau \phi - \epsilon_\alpha \phi^2\right]$$

  with Berry-phase term $\phi\partial_\tau\phi$.
- Quantum phase transitions in metallic or semimetallic systems with fermionic low-energy excitations.
- Topological phase transitions with no local order parameter.

7. Magnetic Quantum Phase Transition

7.1. Order Parameters and Response Functions

Summary

- Observables O are given by an order parameter $\phi(\vec{R})$ and an ordering wavevector $\vec{Q}$ as

$$\langle O(\vec{R})\rangle = \mathrm{Re}[e^{i\vec{Q}\vec{R}}\phi(\vec{R})].$$

- The dynamic structure factor

$$S(\vec{k},\omega) = \int \mathrm{d}^d\vec{r} \int_{-\infty}^{\infty} \mathrm{d}t \langle \phi(\vec{r},t)\phi(0,0)\rangle e^{-i\vec{k}\vec{x}+i\omega t}$$

  and dynamics susceptibility

$$\chi(\vec{k},i\omega_n) = \int \mathrm{d}^d\vec{r} \int_0^{\beta} \mathrm{d}\tau \langle \phi(\vec{r},\tau)\phi(0,0)\rangle e^{-i\vec{k}\vec{x}+i\omega_n\tau}$$

  with ω_n the Matsubara frequencies are related through the fluctuations-dissipation theorem

$$S(\vec{k},\omega) = \frac{2}{1-e^{-\omega/T}} \mathrm{Im}\chi(\vec{k},\omega).$$

The local order parameter ϕ is a function of spatial and temporal coordinates, defined as

$$\langle O(\vec{R})\rangle = \mathrm{Re}[e^{i\vec{Q}\vec{R}}\phi(\vec{R})],$$

where $\phi(\vec{R})$ is a slowly varying function of $\vec{R}$. It can be scalar, vector, tensor, etc. Here, O is some local Hermitian observable, and $\vec{Q}$ is the ordering wavevector, with $e^{i\vec{Q}\vec{R}}$ chosen such that it captures (possible) fast oscillations in $\vec{O}$.

Example 7.1 (Charge density wave, CDW)**.** In a CDW, the order parameter is

$$\langle \rho(\vec{R})\rangle = \rho_0 + \mathrm{Re}[e^{i\vec{Q}\vec{R}}\phi_c(\vec{R})],$$

with $\rho(\vec{R})$ the *charge-density operator*, ρ_0 the *average charge density* and ϕ_c a scalar with $N = 1$ component.

Example 7.2 (Spin density wave, SDW)**.** In a SDW, the order parameter is

$$\vec{S}(\vec{R}) = \mathrm{Re}[e^{i\vec{Q}\vec{R}}\vec{\phi}_s(\vec{R})],$$

where ϕ_s is a vector with $N = 3$ components.

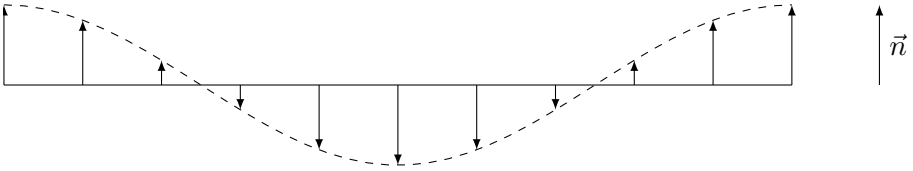
Example 7.3 (Antiferromagnet on a square lattice)**.** The SDW can be used to define the order parameter of an antiferromagnet on a square-lattice with $\vec{Q} = \frac{1}{2}(\vec{b}_1 + \vec{b}_2) = (\frac{\pi}{a}, \frac{\pi}{a})$ and $\vec{\phi}_s$ the "staggered magnetization".

Remark 7.4.
- Vanishing $\vec{Q} = 0$ implies that ϕ can be chosen real for Hermitian O.
- Finite $\vec{Q}$ with $\vec{Q} \neq \sum_{i=1}^{d} \frac{n_i}{2}\vec{b}_i$ implies that ϕ can be complex.

- The phase of complex ϕ is discrete for *commensurate* $\vec{Q}$, i.e., for $\vec{Q} = \sum q_i \vec{b}_i$ with $q_i \in \mathbb{Q}$

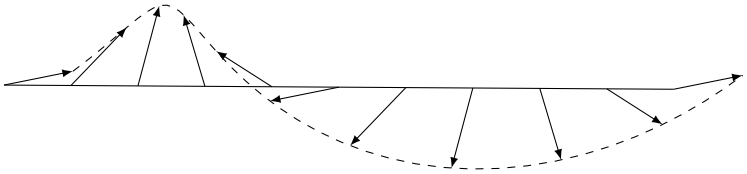
The following classes of spin density waves (vector orders) exist:

- *Collinear* waves, with $\langle \vec{S}_i \rangle \parallel \vec{n}$ for all i implies that $\vec{\phi}_s(\vec{R}) = e^{i\theta(\vec{R})}\vec{n}$



  The $\theta(\vec{R})$ is called the *sliding degree of freedom* and can be used to shift the spin density wave.

- *Spiral* waves, where $\vec{\phi} = \vec{n}_1 + i\vec{n}_2$ with $\vec{n}_1 \cdot \vec{n}_2 = 0$, $\vec{n}_1, \vec{n}_2 \in \mathbb{R}^3$



Definition 7.5 (Real-time correlation function)**.** The *real-time correlation function* is measurable, e.g., through neutron scattering, and defined as

$$C(\vec{r}, t; \vec{r}', t') = \langle \phi(\vec{r}, t)\phi(\vec{r}', t') \rangle.$$

Definition 7.6 (Dynamical structure factor)**.** From this, one may compute the *dynamical structure factor*

$$S(\vec{k}, \omega) = \int \mathrm{d}^d \vec{r} \int_{-\infty}^{\infty} \mathrm{d}t \, C(\vec{r}, t; 0, 0) e^{-i\vec{k}\vec{r} + i\omega t},$$

assuming translational invariance of the correlation function $C(\vec{r}, t; \vec{r}', t') = C(\vec{r} - \vec{r}', t - t'; 0, 0)$.

Definition 7.7 (Imaginary-time correlation function)**.** The *imaginary-time correlation function* is defined as

$$C(\vec{r}, \tau; \vec{r}', \tau') = \langle T_\tau[\phi(\vec{r}, \tau)\phi(\vec{r}', \tau')] \rangle,$$

where τ is imaginary time and

$$T_\tau[A(\tau)B(\tau')] = \begin{cases} A(\tau)B(\tau'), & \tau > \tau', \\ B(\tau')A(\tau), & \tau < \tau'. \end{cases}$$

Definition 7.8 (Dynamical susceptibility)**.** From this, we may define the *dynamical susceptibility*

$$\chi(\vec{k}, i\omega_n) = \int \mathrm{d}^d\vec{r} \int_0^\beta \mathrm{d}\tau \, C(\vec{r}, \tau; 0, 0) e^{-i\vec{k}\vec{r}+i\omega_n\tau}$$

with the (bosonic) Matsubara frequencies $\omega_n = 2\pi nT$, where T is the temperature and $n = 0, \pm 1, \pm 2, \ldots$

Theorem 7.9 (Fluctuation-dissipation theorem)**.** *The dynamical susceptibility is related to the dynamical structure factor, i.e., it is possible to move from imaginary-time to real-time, such that*

$$S(\vec{k}, \omega) = \frac{2}{1 - e^{-\omega/T}} \mathrm{Im}\chi(\vec{k}, \omega),$$

with $\mathrm{Im}\chi(\vec{k}, \omega)$*, where the practical problem is in the analytical continuation of* $\chi(\vec{k}, i\omega)$ *to the real axis, i.e.,* $i\omega_n \leftrightarrow \omega + i\delta|_{\delta\to 0}$.

7.2. Properties of the Quantum ϕ^4 Model

Consider the action

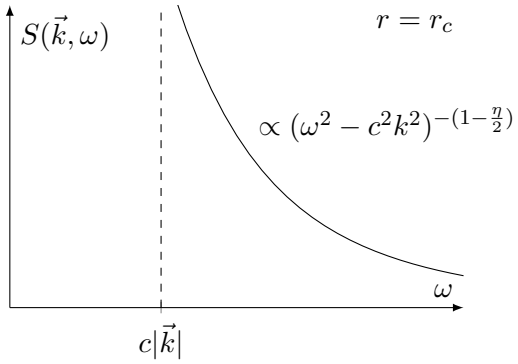
$$S = \int \mathrm{d}^d\vec{r} \int_0^\beta \mathrm{d}\tau \left(\frac{c^2}{2} (\nabla\phi)^2 + \frac{1}{2} (\partial_\tau \phi)^2 + \frac{r}{2}\phi^2 + \frac{u}{4!}\phi^4 \right).$$

It has the following zero-temperature properties:

- Dynamical exponent $z = 1$.
- Quantum-to-classical correspondence: QCP$(d)\hat{=}$TCP$(d+1)$.
- Critical two-point correlator

  $$\chi(\vec{k}, \omega) \propto \frac{1}{[c^2\vec{k}^2 - (w + i\delta)^2]^{\frac{2-\eta}{2}}},$$

  which corresponds to the *critical continuum of excitations* with $\omega + i\delta|_{\delta\to 0} \leftrightarrow i\omega_n$.
- Dynamical structure factor (from the dissipation fluctuation theorem) $S(\vec{k}, \omega) \propto \mathrm{Im}\chi(\vec{k}, \omega)$, having no quasiparticle excitations for $\eta \neq 0$.



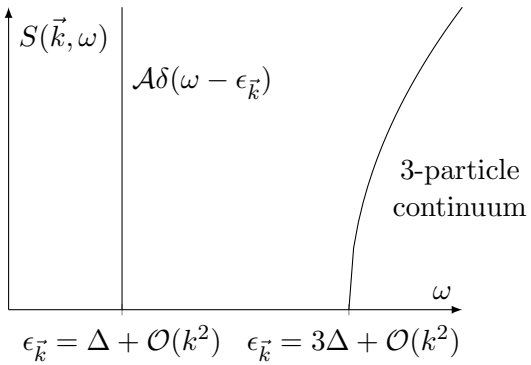
Disordered Phase ($r > r_c$). In the disordered phase, we have

$$S(\vec{k},\omega) \propto \mathrm{Im}\chi(\vec{k},\omega) = \mathrm{Im}\left[\frac{1}{c^2\vec{k}^2 + r - (w+i\delta)^2 - \Sigma(\vec{k},\omega)}\right]$$
$$= \frac{\mathcal{A}}{2\epsilon_{\vec{k}}}\left[\delta(\omega - \epsilon_{\vec{k}}) - \delta(\omega + \epsilon_{\vec{k}})\right],$$

for small ω with quasiparticle dispersion $\epsilon_{\vec{k}}^2 = c^2k^2 + r - \Sigma$, Σ the self-energy from perturbation theory in the $\frac{u}{4!}\phi^4$ term and quasiparticle pole $\mathcal{A}$.

Remark 7.10.

- Self-energy Σ modifies $\epsilon_{\vec{k}}$ and $\mathcal{A}$, but does *not* remove quasiparticle pole.
- Energy gap $\Delta = \sqrt{r - \Sigma(\vec{k} = 0, \omega = \Delta)}$ is the frequency of quasiparticles at $\vec{k} = 0$.
- Dispersion near $k = 0$ is $\epsilon_{\vec{k}} = \Delta + \frac{c^2}{2\Delta}k^2 + \mathcal{O}(k^4)$.
- Higher energies with $\omega > n\Delta$, $n = 3, 5, 7, \ldots$ are the n-particle continua.



- In the limit $r \searrow r_c$

$$\chi(\vec{k},\omega) = \frac{1}{\Delta^{2-\eta}} f_\chi\left(\frac{ck}{\Delta}, \frac{\omega}{\Delta}\right),$$

implying that the single-particle pole needs to go to zero as $\mathcal{A} \propto \Delta^\eta \propto (r - r_c)^{\eta\nu}$ $(z = 1)$

Ordered Phase ($r < r_c$). In the ordered phase, we have

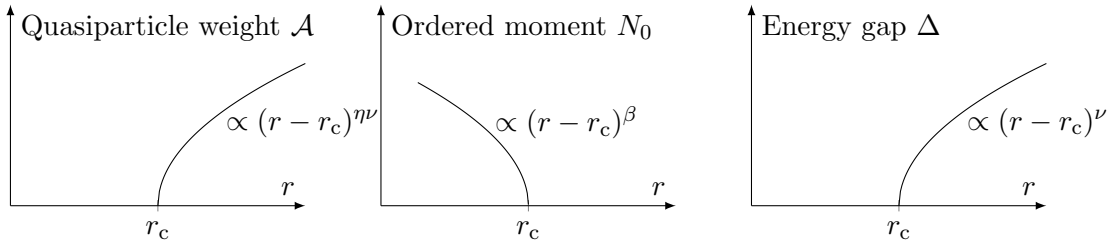
$$S(\vec{k}, \omega) = N_0^2 (2\pi)^{d+1} \delta(\omega)\delta(\vec{k}) + \mathcal{O}(\omega)$$

with N_0 the order parameter. The spin-structure factor exhibits the *Bragg peak* at $\vec{k} = 0$.

The susceptibility ($N > 1$) has the following features:

- transverse susceptibility $\chi_\perp$ with poles at $\omega = 0$ called *Goldstone modes*,
- longitudinal susceptibility $\chi_\parallel$ with poles at $\omega \neq 0$ called *Higgs mode*.

In summary



7.3. Quantum Ising Chain

Summary

- The quantum Ising chain can be diagonalized in 1D using Jordan-Wigner transformation and subsequent Bogoliubov transformation.

The Hamiltonian of the Quantum Ising chain (after Jordan-Wigner transformation) is

$$H_\mathrm{I} = -J \sum_i (c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i + c_{i+1}^\dagger c_i - 2g c_i^\dagger c_i + g),$$

which after Fourier-transform turns into

$$H_\mathrm{I} = J \sum_k \left[2(g - \cos(ka)) c_k^\dagger c_k - i \sin(ka)(c_{-k}^\dagger c_k^\dagger + c_{-k} c_k) - g \right].$$

To remove the anomalous terms, we perform the Bogoliubov transformation

$$\gamma_k = u_k c_k - i v_k c_{-k}^\dagger \quad \Longleftrightarrow \quad c_k = u_k \gamma_k + i v_k \gamma_{-k}^\dagger,$$

with $u_k^2 + v_k^2 = 1$ such that the γ_k satisfy fermionic commutation relations. We parameterize $u_k = \sin\frac{\theta_k}{2}$ and $v_k = \cos\frac{\theta_k}{2}$. For

$$\tan\theta_k = \frac{\sin ka}{\cos ka - g}$$

the anomalous terms cancel and the Hamiltonian becomes diagonal in the new basis.

Proof. In the following, we ignore the $k = 0$ mode, for which the above is straightforward to show, as the Hamiltonian is already diagonal. Furthermore, we already use $\{\gamma_k, \gamma_{k'}^\dagger\} = \delta_{kk'}$ and the parameterization of u_k and v_k with $u_k = -u_{-k}$ and $v_k = v_{-k}$. We may then compute the contribution of the first term for the remaining modes as

$$c_k^\dagger c_k = u_k^2 \gamma_k^\dagger \gamma_k - i v_k u_k (\gamma_{-k}\gamma_k + \gamma_{-k}^\dagger \gamma_k^\dagger) + v_k^2 \gamma_{-k}\gamma_{-k}^\dagger.$$

For the second term, we obtain

$$\begin{aligned} c_{-k}^\dagger c_k^\dagger &= -u_k^2 \gamma_{-k}^\dagger \gamma_k^\dagger + i v_k u_k (\gamma_{-k}^\dagger \gamma_{-k} - \gamma_k \gamma_k^\dagger) - v_k^2 \gamma_k \gamma_{-k}, \\ c_{-k} c_k &= -u_k^2 \gamma_{-k}\gamma_k + i v_k u_k (\gamma_k^\dagger \gamma_k - \gamma_{-k}\gamma_{-k}^\dagger) - v_k^2 \gamma_k^\dagger \gamma_{-k}^\dagger. \end{aligned}$$

Their sum is

$$c_{-k}^\dagger c_k^\dagger + c_{-k} c_k = -(u_k^2 - v_k^2)(\gamma_{-k}^\dagger \gamma_k^\dagger + \gamma_{-k}\gamma_k) + 2 i v_k u_k (\gamma_{-k}^\dagger \gamma_{-k} + \gamma_k^\dagger \gamma_k).$$

If all mixed terms are to cancel one another, we require that $-2(g - \cos ka) i v_k u_k + i \sin ka (u_k^2 - v_k^2) = 0$. Note that $u_k v_k = \frac{\sin\theta_k}{2}$ and $u_k^2 - v_k^2 = -\cos\theta_k$. Therefore, we require

$$\tan\theta_k = \frac{\sin\theta_k}{\cos\theta_k} = \frac{\sin ka}{\cos ka - g}.$$

For the remaining Hamiltonian, we have

$$\begin{aligned} H_\mathrm{I} &= J \sum_k \left[2(g - \cos ka))(u_k^2 \gamma_k^\dagger \gamma_k + v_k^2 \gamma_{-k}\gamma_{-k}^\dagger) + 2 v_k u_k \sin ka (\gamma_{-k}^\dagger \gamma_{-k} + \gamma_k^\dagger \gamma_k) \right] \\ &= \sum_k (2J\sqrt{1 + g^2 - 2g\cos ka}) \gamma_k^\dagger \gamma_k. \end{aligned}$$

□

We have

$$H_\mathrm{I} = \sum_k \epsilon_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right)$$

with dispersion $\epsilon_k = 2J\sqrt{1 + g^2 - 2g\cos ka} \geq 0$. The *band gap* scales as

$$\Delta = \epsilon_{k=0} = 2J\,|1 - g| = 0 \iff g = 1 = g_\mathrm{c}.$$

Its *critical behavior* is

$$\Delta \propto \frac{1}{\tau_c} \propto \left(\frac{1}{\xi}\right)^z \propto |g - g_c|^{\nu z},$$

implying that $\nu z = 1$.

Further results from the exact solution. From the results shown in Fig. 12, we find $\beta = \frac{1}{8}$ and $\eta = \frac{1}{4}$.

We obtain the finite-temperature phase diagram depicted in Fig. 13.

8. Quantum Phase Transition of Bosons and Fermions

8.1. Bose-Hubbard Model

Summary

- The Hamiltonian of the Bose-Hubbard model is

$$H = -w \sum_{\langle i,j \rangle} (b_i^\dagger b_j + b_j^\dagger b_i) - \mu \sum_i n_i + \frac{1}{2} U \sum_i n_i (n_i - 1).$$

- For no interaction $U = 0$, the bosons condensate, while for small interaction $U \ll w$, they form a superfluid. For large interactions $U \gg w$ a Mott insulating state can form for integer filling.

The Hamiltonian is

$$H = -w \sum_{\langle i,j \rangle} (b_i^\dagger b_j + b_j^\dagger b_i) - \mu \sum_i n_i + \frac{1}{2} U \sum_i n_i (n_i - 1),$$

with w the *hopping parameter*, μ the *chemical potential*, $n_i = b_i^\dagger b_i$ the *number operator*, U the *onsite potential* and $[b_i, b_j^\dagger] = \delta_{ij}$.

This model exhibits the following low-temperature ($k_B T \ll w$) phases:

1. $U/w = 0$ leads to non-interacting Bosons forming a *Bose-Einstein condensate* with all particles in the single-particle ground state ($d > 2$).
2. $0 < U/w \ll 1$ leads to a *superfluid* with $\langle b_i \rangle \neq 0$.
3. $U/w \gg 1$ leads to a *Mott insulator* for integer filling ($\langle n \rangle = \frac{1}{M} \sum_i \langle n_i \rangle \in \mathbb{N}$), because all hopping is highly suppressed, and to a *superfluid* for non-integer filling ($\langle n \rangle \in \mathbb{N} + \delta n$), because the energy penalty for two bosons on the same site is already incurred.

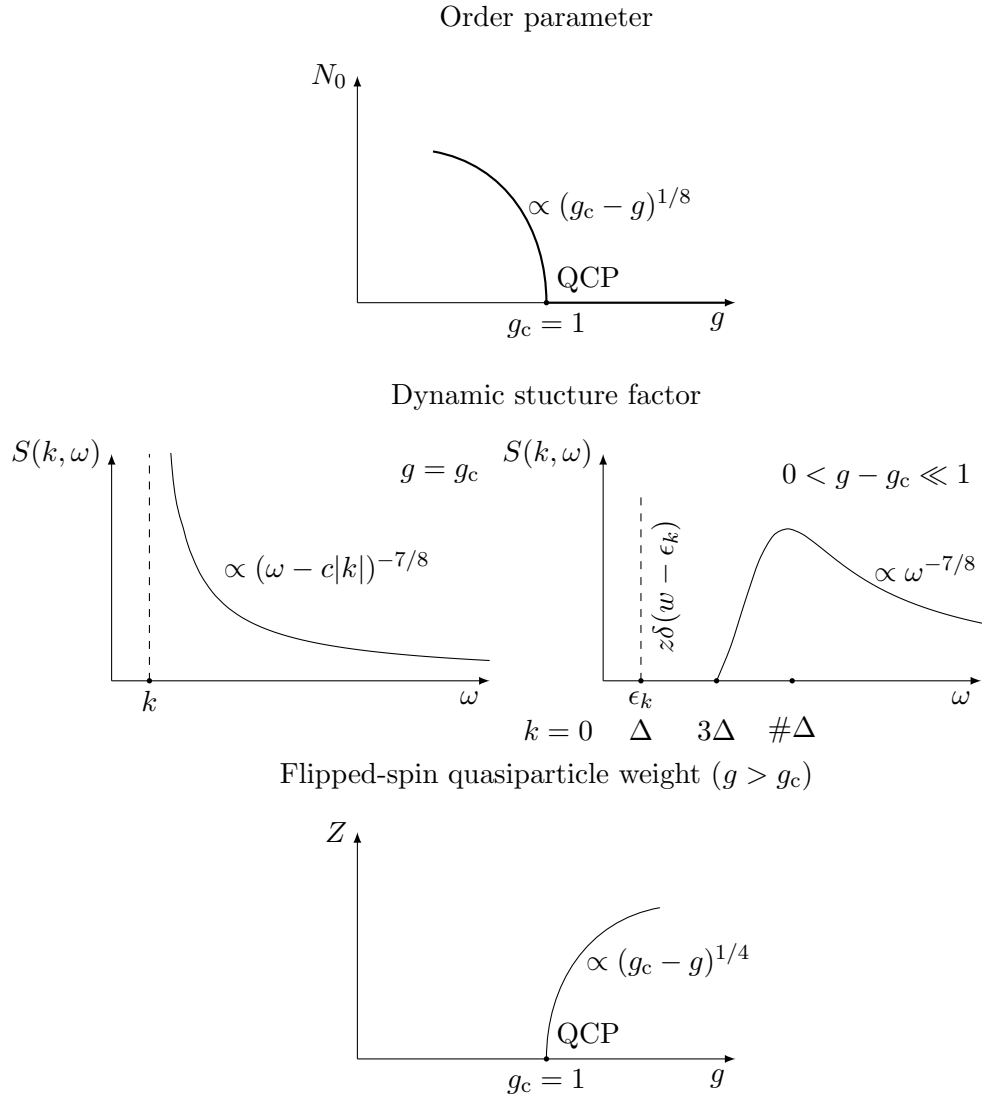


Figure 12: Further results of the exact solution of the quantum Ising model, yielding $\beta = 1/8$ from the order parameter scaling, $\eta = 1/4$ from the dynamic structure factor and $\eta\nu = 1/4$ from the flipped-spin quasiparticle weight.

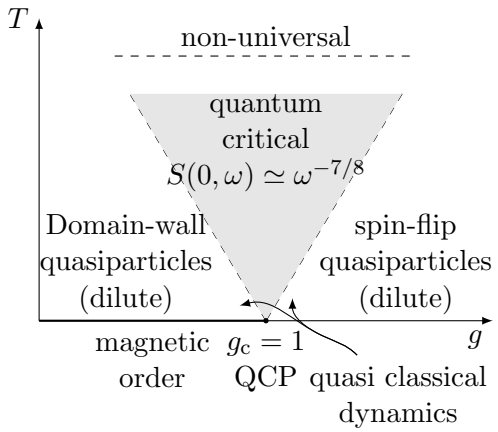


Figure 13: Finite-T phase diagram of the quantum Ising chain.

Experimental realizations are ultracold atoms on an optical lattice and magnons at a field-driven QCP ($\mu \leftrightarrow B$).

8.2. Mean-Field Theory

Summary

- The mean-field order parameter $\Psi_{\rm B} = zw\langle b_i\rangle$ vanishes for localized bosons in the Mott insulating phase and spontaneously breaks the U(1) symmetry in the superfluid phase.

Define the order-parameter

$$\Psi_{\rm B} = zw\langle b_i\rangle, \qquad b_i = \frac{1}{zw}\Psi_{\rm B} + \delta b_i,$$

with z the *lattice coordination number*. Upon mean-field decoupling the in kinetic term, we obtain

$$H_{\rm MF} = \sum_i \left[-\mu n_i + \frac{1}{2}Un_i(n_i-1) - \Psi_{\rm B}^* b_i - \Psi_{\rm B} b_i^\dagger\right].$$

This Hamiltonian is local.

The possible phases of the mean-field Hamiltonian are

- a *superfluid* for $\Psi_B \neq 0$ with U(1) symmetry, i.e., $\Psi_B = |\Psi_B|\, e^{i\phi} \mapsto |\Psi_B|\, e^{i(\phi+\delta\phi)}$, spontaneously broken, and
- a *Mott insulator* for $\Psi_B = 0$ with localized bosons and no long-range order.

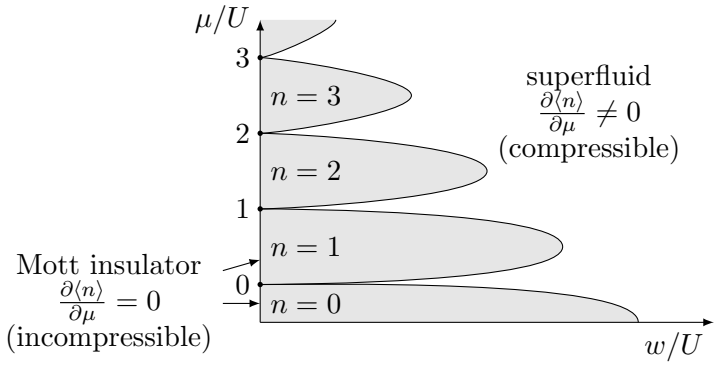


Figure 14: Mean-field phase diagram of the Bose-Hubbard model.

For $w \to 0$, the mean-field groundstate is $\Psi_{\mathrm{B}} = 0$ with

$$n = \langle n_i \rangle = \begin{cases} 0 & \text{for } \mu < 0 \\ 1 & \text{for } 0 < \mu/U < 1 \\ 2 & \text{for } 1 < \mu/U < 2 \\ \vdots & \end{cases}$$

Remark 8.1. The groundstate for finite w can be computed in standard quantum mechanical perturbation theory.

Figure 14 shows the mean-field phase diagram of the Bose-Hubbard model.

8.3. Superfluid-Insulator Transition - Universality Classes

Summary

- The phase transition in the Bose-Hubbard model can be studied using the coherent-state path integral with bosonic variables Φ_i and Φ_i^* with action

$$S_{\mathrm{B}} = \sum_i \int_0^\beta \mathrm{d}\tau \left[\Phi_i^* \frac{\partial \Phi_i}{\partial \tau} - \mu \Phi_i^* \Phi_i + \frac{U}{2} \Phi_i^* \Phi_i (\Phi_i^* \Phi_i - 1) \right] - w \sum_{\langle ij \rangle} \int_0^\beta \mathrm{d}\tau (\Phi_i^* \Phi_j + \Phi_j^* \Phi_i).$$

- The Hubbard-Stratonovich transformation allows to decouple the local fields,

by introducing the additional fields $\Psi_{\mathrm{B}i}$ with action

$$S'_{\mathrm{B}} = \int_0^\beta \mathrm{d}\tau \Big[\sum_i \left(\Phi_i^* \frac{\partial \Phi_i}{\partial \tau} - \mu \Phi_i^* \Phi_i + \frac{1}{2} U \Phi_i^* \Phi_i (\Phi_i^* \Phi_i - 1) - \Psi_{\mathrm{B}i} \Phi_i^* - \Psi_{\mathrm{B}i}^* \Phi_i \right) + \sum_{i,j} \Psi_{\mathrm{B}i} w_{ij}^{-1} \Psi_{\mathrm{B}i} \Big].$$

- Symmetry arguments yield the form of the action after integrating the Φ_i as

$$S''_{\mathrm{B}} = \int \mathrm{d}^d \vec{x} \int_0^\beta \mathrm{d}\tau \left(K_1 \Psi_{\mathrm{B}}^* \frac{\partial \Psi_{\mathrm{B}}}{\partial \tau} + K_2 \left| \frac{\partial \Psi_{\mathrm{B}}}{\partial \tau} \right|^2 + r \left| \Psi_{\mathrm{B}} \right|^2 + K_3 \left| \nabla \Psi_{\mathrm{B}} \right|^2 + \frac{U}{2} \left| \Psi_{\mathrm{B}} \right|^4 + \dots \right)$$

- From the U(1) symmetry of the Bose-Hubbard model, it follows that $K_1 = -\partial r / \partial \mu$. Hence, if the transition is not tunable by changing μ, then $K_1 = 0$.
  - The transition through the tip of the lobe has $K_1 = 0$ and therefore $z = 1$.
  - Otherwise, the transition has $K_1 \neq 0$ and therefore $z = 2$.

In the superfluid phase, there are lines of constant density, as shown in Fig. 15. In the following, the universality classes of the phase transitions are derived.

The coherent-state path integral

$$Z_{\mathrm{B}} = \int \mathcal{D}\Phi_i(\tau) \mathcal{D}\Phi_i^*(\tau) e^{-S_{\mathrm{B}}[\Phi_i(\tau), \Phi_i^*(\tau)]}$$

with action

$$S_{\mathrm{B}} = \sum_i \int_0^\beta \mathrm{d}\tau \left[\Phi_i^* \frac{\partial \Phi_i}{\partial \tau} - \mu \Phi_i^* \Phi_i + \frac{U}{2} \Phi_i^* \Phi_i (\Phi_i^* \Phi_i - 1) \right] - w \sum_{\langle ij \rangle} \int_0^\beta \mathrm{d}\tau (\Phi_i^* \Phi_j + \Phi_j^* \Phi_i).$$

We perform the Hubbard-Stratonovich transformation

$$Z_{\mathrm{B}} = \int \mathcal{D}\Phi_i \mathcal{D}\Phi_i^* \mathcal{D}\Psi_{\mathrm{B}i} \mathcal{D}\Psi_{\mathrm{B}i}^* e^{-S_{\mathrm{B}'}[\Phi_i, \Phi_i^*, \Psi_{\mathrm{B}i}, \Psi_{\mathrm{B}i}^*]}$$

with action

$$S'_{\mathrm{B}} = \int_0^\beta \mathrm{d}\tau \Big[\sum_i \left(\Phi_i^* \frac{\partial \Phi_i}{\partial \tau} - \mu \Phi_i^* \Phi_i + \frac{1}{2} U \Phi_i^* \Phi_i (\Phi_i^* \Phi_i - 1) - \Psi_{\mathrm{B}i} \Phi_i^* - \Psi_{\mathrm{B}i}^* \Phi_i \right) + \sum_{i,j} \Psi_{\mathrm{B}i} w_{ij}^{-1} \Psi_{\mathrm{B}i} \Big]$$

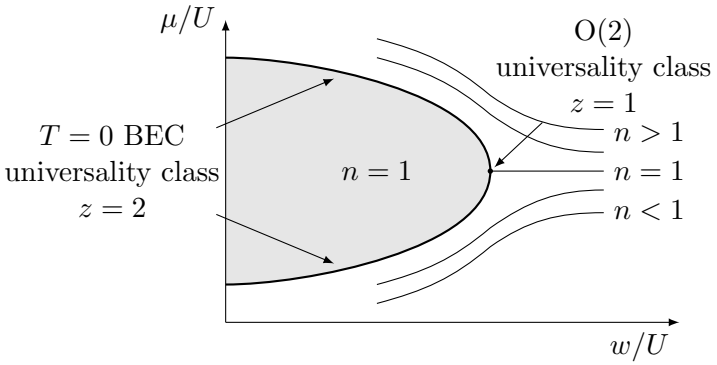


Figure 15: One Mott-lobe with the lines of constant density in the superfluid phase. In the non-interacting limit $w/U \gg 1$ the lines must be evenly spaced, as the particle density should increase linearly with the chemical potential. The tip of the Mott-lobe merges with line at integer n, because there the least energy penalty needs to be overcome.

and *hopping matrix*

$$w_{ij} = \begin{cases} w, & \text{if } ij \text{ are neighbors,} \\ 0, & \text{otherwise.} \end{cases}$$

Integrating over Φ_i and Φ_i^* yields

$$Z_{\mathrm{B}} = \int \mathcal{D}\Psi_{\mathrm{B}} \mathcal{D}\Psi_{\mathrm{B}}^* e^{-S'_{\mathrm{B}}[\Psi_{\mathrm{B}}, \Psi_{\mathrm{B}}^*]}$$

with *effective action* (continuum limit $\Psi_{\mathrm{B}i}(\tau) \mapsto \Psi_{\mathrm{B}}(\tau, \vec{x})$). We obtain

$$S''_{\mathrm{B}} = \int \mathrm{d}^d \vec{x} \int_0^\beta \mathrm{d}\tau \left(K_1 \Psi_{\mathrm{B}}^* \frac{\partial \Psi_{\mathrm{B}}}{\partial \tau} + K_2 \left| \frac{\partial \Psi_{\mathrm{B}}}{\partial \tau} \right|^2 + r \left| \Psi_{\mathrm{B}} \right|^2 + K_3 \left| \nabla \Psi_{\mathrm{B}} \right|^2 + \frac{U}{2} \left| \Psi_{\mathrm{B}} \right|^4 + \dots \right)$$

by symmetry alone.

Using the "simple" U(1) symmetry restricted to the temporal variable τ, we relate

$$\begin{aligned} \Phi_i &\mapsto \Phi_i e^{i\phi(\tau)} \\ \Psi_{\mathrm{B}i} &\mapsto \Psi_{\mathrm{B}i} e^{i\phi(\tau)} \\ \mu &\mapsto \mu + i \frac{\partial \phi}{\partial \tau} \end{aligned}$$

with the "gauge field" $\phi(\tau)$. Now, we demand the gauge invariance of S''_{B} and obtain the relation

$$K_1 = -\frac{\partial r}{\partial \mu}.$$

Proof. Schematically, the Lagrangian is

$$\mathcal{L} = K_1 \Psi_{\mathrm{B}}^* \frac{\partial}{\partial \tau} \Psi_{\mathrm{B}} + r \left|\Psi_B\right|^2 ,$$

in which $r(\mu + \delta\mu) = r(\mu) + \frac{\partial r}{\partial \mu}\delta\mu + \ldots$ may depend on μ with $\delta\mu = i\frac{\partial \phi}{\partial \tau}$. Performing the gauge transformation $\Psi_B \mapsto \Psi_B e^{i\phi}$ and $r \mapsto r + i\frac{\partial r}{\partial \mu}\frac{\partial \phi}{\partial \tau}$, we find

$$\mathcal{L} = K_1 \Psi_{\mathrm{B}}^* \frac{\partial}{\partial \tau} \Psi_{\mathrm{B}} + i K_1 \left|\Psi_{\mathrm{B}}\right|^2 \frac{\partial \phi}{\partial \tau} + r \left|\Psi_{\mathrm{B}}\right|^2 + i \frac{\partial r}{\partial \mu}\frac{\partial \phi}{\partial \tau} \left|\Psi_{\mathrm{B}}\right|^2 .$$

For this to be gauge invariant, it has to hold that

$$K_1 = -\frac{\partial r}{\partial \mu}.$$

□

This implies that either r depends on μ and changing μ allows to tune the model, or r is independent of μ, hence $K_1 = 0$ and we cannot tune the system using μ. In other words, the K_1 term is present, if the transition can be tuned using μ. Comparing quadratic and linear time derivative, we find

(a) Density $\langle n \rangle$ is fixed across transitions

- Transition through tip of "Mott lobe", tunable by w only
- $K_1 = 0 \implies z = 1$
- O(2) universality class in $d + 1$ dimensions

(b) Density $\langle n \rangle$ varies across the transition

- Transition is tunable using μ
- $K_1 \neq 0 \implies z = 2$ (in the non-interacting limit)
- $T = 0$ BEC universality class with $\eta = 0$ and $\nu = \frac{1}{2}$ (fully quantum, without classical analogue)

8.4. Dilute Bose Gas

Summary

- The dilute Bose gas can be utilized to understand the BEC transition and has action

$$S_{\mathrm{B}} = \int \mathrm{d}\tau \int \mathrm{d}^d \vec{x} \left(\Phi^* \frac{\partial \Phi}{\partial \tau} + \frac{1}{2m} \left|\nabla \Phi\right|^2 - \mu \left|\Phi\right|^2 + \frac{U}{2} \left|\Phi\right|^4 \right).$$

- For $\mu < 0$ it has no particle density $\langle \Phi \rangle = 0$, while for $\mu > 0$ it is in a superfluid

phase with $\langle\Phi\rangle \neq 0$. The point $\mu = 0$ described the Mott-superfluid transition.

To further understand the BEC universality class, we investigate this transition as it occurs in the dilute Bose gas. The Hamiltonian is

$$H_{\mathrm{B}} = \sum_{\vec{k}} \frac{\vec{k}^2}{2m} b_{\vec{k}}^\dagger b_{\vec{k}} - \sum_i \mu n_i + \frac{1}{2}\sum_i U n_i(n_i - 1)$$

and corresponding action in the continuum limit

$$S_{\mathrm{B}} = \int \mathrm{d}\tau \int \mathrm{d}^d\vec{x} \left(\Phi^* \frac{\partial \Phi}{\partial \tau} + \frac{1}{2m} |\nabla\Phi|^2 - \mu |\Phi|^2 + \frac{U}{2} |\Phi|^4 \right).$$

The phases at $T = 0$ are

(a) $\mu < 0$: $\langle\Phi\rangle = 0$ with density $\langle\Phi^*\Phi\rangle = 0$, i.e., no particles.

(b) $\mu > 0$: $\langle\Phi\rangle \neq 0$ superfluid with $\langle\Phi^*\Phi\rangle \neq 0$.

The QCP at $\mu = 0$ describes the Mott-superfluid transition in the Bose-Hubbard model, modulo the integer background density.

The RG flow near $\mu = 0$ is

$$\frac{\mathrm{d}u}{\mathrm{d}\ln b} = (4 - d - z)u - \frac{u^2}{2}, \qquad \text{with } u = \frac{S_d}{(2\pi)^d} \frac{2mU}{\Lambda^{2-d}},$$
$$\frac{\mathrm{d}\tilde{\mu}}{\mathrm{d}\ln b} = 2\tilde{\mu}, \qquad \text{with } \tilde{\mu} = \frac{\mu}{\Lambda^2},$$

with $\eta = 0$ and $z = 2$. We find the following critical behavior:

- Above the upper critical dimension $d \geq d_c^+ = 2$,
  - u is (dangerously) irrelevant, and the system is at Gaussian criticality with $z = 2$, $\eta = 0$ and $\nu = \frac{1}{2}$.
  - The density of the bosons is

$$\langle\Phi^*\Phi\rangle = \begin{cases} 0, & \mu < 0, \\ \frac{\mu}{u} + \mathcal{O}(\mu^2), & \mu > 0. \end{cases}$$

- Below the upper critical dimension $d = 1$,
  - u is relevant, the critical behavior is governed by a non-Gaussian fixed point with $z = 2$, and still $\eta = 0$ and $\nu = \frac{1}{2}$.
  - It is equivalent to dilute Fermi gas QCP in 1D.

The phase diagram of the dilute Bose gas is depicted in Fig. 16.

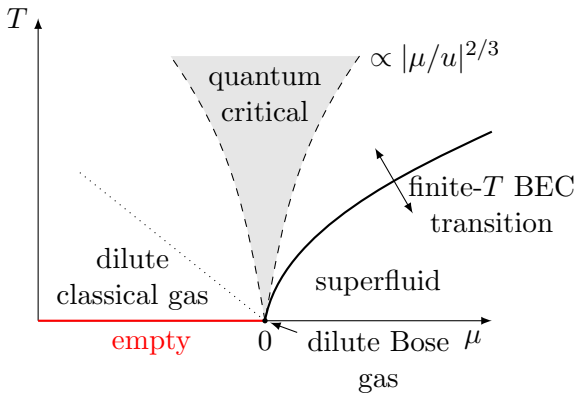


Figure 16: Phase diagram of the dilute Bose gas in $d = 3$ dimensions. In the empty area, there are no particles.

8.5. Dilute Spinless Fermi Gas

Summary

- The action of the dilute spinless Fermi gas is

$$S_{\mathrm{F}} = \int \mathrm{d}\tau \int \mathrm{d}^d\vec{x} \left(\psi^* \frac{\partial \psi}{\partial \tau} + \frac{1}{2m} |\nabla\psi|^2 - \mu |\psi|^2 \right).$$

- The critical exponents are the same as in the 1d Bose gas.

Hamiltonian for free fermions

$$H_{\mathrm{F}} = \sum_{\vec{k}} \frac{\vec{k}^2}{2m} c^\dagger_{\vec{k}} c_{\vec{k}} - \sum_i \mu c^\dagger_i c_i$$

with $\{c_i, c^\dagger_j\} = \delta_{ij}$. The action is

$$S_{\mathrm{F}} = \int \mathrm{d}\tau \int \mathrm{d}^d\vec{x} \left(\psi^* \frac{\partial \psi}{\partial \tau} + \frac{1}{2m} |\nabla\psi|^2 - \mu |\psi|^2 \right)$$

where ψ and ψ^* are Grassmann variables, $\psi^2 = (\psi^*)^2 = 0$.

Remark 8.2.

- Any on-site interaction impossible, as $|\psi|^4 = 0$.
- Non-local interactions, such as $|\psi|^2 |\nabla\psi|^2$ are always irrelevant.

The particle density is

$$\langle \psi^* \psi \rangle = \begin{cases} 0, & \text{for } \mu < 0, \\ \frac{S_d}{(2\pi)^d} \frac{(2m\mu)^{d/2}}{d}, & \text{for } \mu > 0, \end{cases}$$

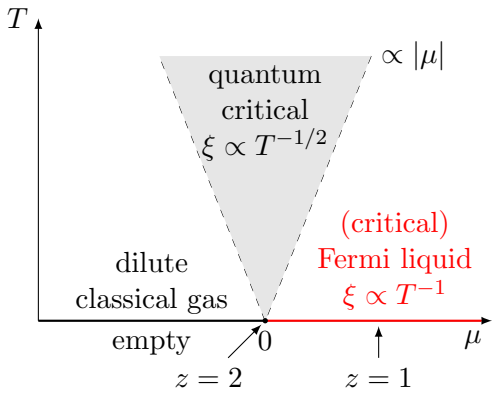


Figure 17: Phase diagram of the dilute spinless Fermi gas.

with $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ the surface area of the hypersphere. This transition is also called *Lifshitz transition* and is related to a change of the topology of the Fermi sphere. The free energy density is

$$f_{\mathrm{F}} = -T \int \frac{\mathrm{d}^d \vec{k}}{(2\pi)^d} \ln \left[1 + e^{-\left(\mu - \frac{\vec{k}^2}{2m}\right)/T} \right].$$

We substitute $\vec{k} = \frac{k}{\sqrt{T}}$ and $\mathrm{d}^d \vec{k} = T^{-d/2} \mathrm{d}^d \vec{k}$ finding

$$f_{\mathrm{F}} = T^{\frac{d+2}{2}} \tilde{\Phi}_{\mathrm{F}} \left(\frac{\mu}{T} \right) = \mu^{\frac{d+2}{2}} \Phi_{\mathrm{F}} \left(\frac{T}{\mu} \right)$$

with universal scaling functions $\tilde{\Phi}_{\mathrm{F}}$ and Φ_{F}. This yields the critical exponents

$$f_{\mathrm{F}}(\mu \to 0) \propto T^{\frac{d+z}{z}} \implies z = 2, \qquad f_{\mathrm{F}}(T \to 0) \propto |\mu|^{\nu(d+z)} \implies \nu = \frac{1}{2}.$$

The phase diagram of the dilute spinless Fermi gas is depicted in Fig. 17.

8.6. Fermi-Hubbard Model on the Square Lattice

The model is described by the Hamiltonian

$$H = -t \sum_{\langle ij \rangle, \sigma} (c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma}) - \mu \sum_{i,\sigma} n_{i\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}.$$

The simplified phase diagram of the Fermi-Hubbard model on the square lattice is depicted in Fig. 18.

There is antiferromagnetism in the Mott insulator.

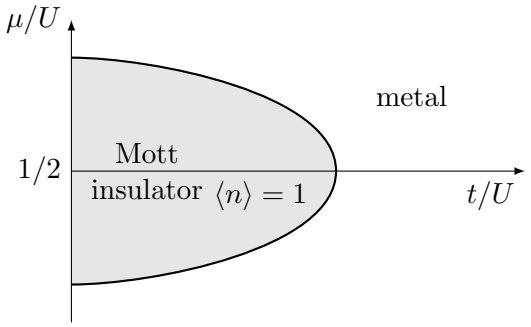


Figure 18: Simplified phase diagram of the Fermi-Hubbard model on the square lattice.

(a) Without hopping, i.e., at $t/U = 0$, the spins are localized, and the groundstate is 2^N-fold degenerate.

(b) For small hoppings ($0 < t/U \ll 1$) virtual hopping processes lead to an energy gain of antiferromagnetic states.

- The effective model is

$$H = J \sum_{\langle ij \rangle} \left(S_j S_j - \frac{1}{4} \right)$$

with $J = 4\frac{t^2}{u}$ and $S_i = c_{i\sigma} \tau_{\sigma\sigma'} c_{i\sigma}$

Remark 8.3. The phase diagram is not fully understood, but presumably rich with magnetic phases, superconducting phases, nematic phases, spin liquid phases, etc.

8.7. Fermi-Hubbard Model on the Honeycomb Lattice

Spinless fermions with nearest-neighbor repulsion

$$H = -t \sum_{\langle ij \rangle} (a_i^\dagger b_j + b_j^\dagger a_i) + V \sum_{\langle ij \rangle} a_i^\dagger a_i b_j^\dagger b_j.$$

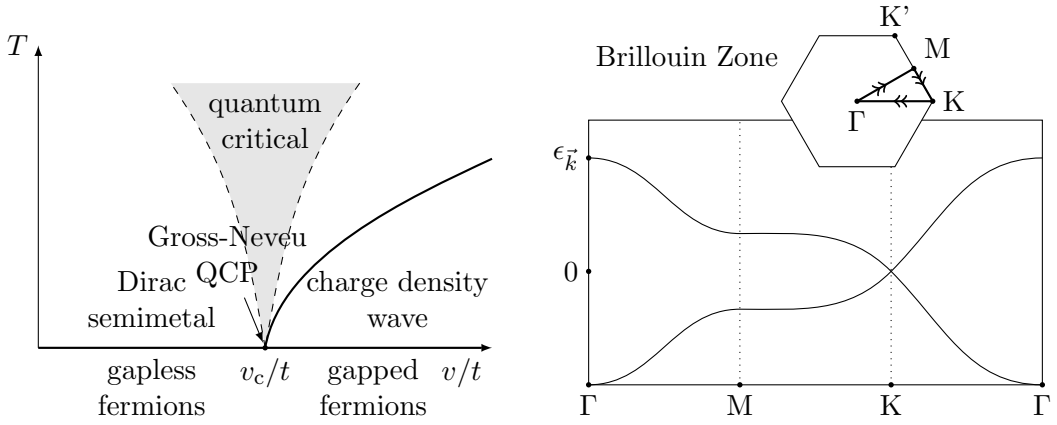
The energy dispersion relation and phase diagram are depcited in Fig. 19.

The Gross-Neveu point may be described by the effective model

$$S = \int \mathrm{d}\tau \int \mathrm{d}^2 \vec{x} \left[\bar{\Psi} \gamma_\mu \partial_\mu \Psi + g \phi \bar{\Psi} \Psi + \frac{1}{2}(-\partial_\mu^2 + r)\phi + \lambda \phi^4 \right]$$

with

$$\Psi(\vec{q}, \omega) = \begin{pmatrix} a(\vec{K} + \vec{q}, \omega) \\ b(\vec{K} + \vec{q}, \omega) \\ a(-\vec{K} + \vec{q}, \omega) \\ b(-\vec{K} + \vec{q}, \omega) \end{pmatrix}$$



(a) Phase diagram.

(b) Energy dispersion.

Figure 19: Phase diagram at $\mu = 0$ and energy disperion at $V = 0$ of the Fermi-Hubbard model on the honeycomb lattice.

and $\bar{\Psi} = \Psi^\dagger \gamma_0$ with $\{\gamma_0, \gamma_\nu\} = 2\delta_{\mu\nu} I$, and $\mu, \nu = 0, 1, 2$.

The critical behavior obtained from $1/N$ expansion, ϵ expansion, QMC, and conformal bootstrap is described by the critical exponents

$$\eta_\phi \simeq 0.51, \qquad \eta_\Psi \simeq 0.087, \qquad \nu \simeq 0.91, \qquad z = 1,$$

which define the 2+1d Gross-Neveu universality class.

Part II.
Exercises

9. Landau functional for a first-order phase transition

Consider the free-energy density

$$f(\phi) = \frac{a}{2}\phi^2 + \frac{b}{4}\phi^4 + \frac{c}{6}\phi^6, \tag{1}$$

which depends on the real order parameter ϕ. The parameter $a = a(T)$ depends on the temperature T, the coefficients b and c are temperature-independent constants, and $b < 0$, $c > 0$.

(a) Determine the extrema of the functional $f(\phi)$ in Eq. (1). List all possibilities and sketch $f(\phi)$ in each case.

The first derivative of the free-energy density functional is

$$f'(\phi) = \phi(a + b\phi^2 + c\phi^4),$$

yielding the extrema at

$$\phi_0 = 0, \qquad \phi_\pm^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}.$$

While ϕ_0 is always a valid solution, the $\phi_\pm^2$ may turn negative or imaginary, corresponding to vanishing extrema for real $f(\phi)$. The respective conditions are

$$b^2 > 4ac,$$

such that both $\phi_\pm^2$ remain real and

$$a > 0,$$

for ϕ_-^2 to remain positive. The respective cases are sketched in Fig. 20.

(b) Calculate the critical value a_c of the parameter a where the position $\phi_\mathrm{eq}(a)$ of the global minimum of $f(\phi)$ changes discontinuously.

The critical value may be obtained when the global minima is at ϕ_0 and ϕ_+^2 becomes a local minima. The two minima are degenerate global minima, when $f(\phi_+^2, a_\mathrm{c}) = 0$, representing the critical value a_c. This expression may be evalu-

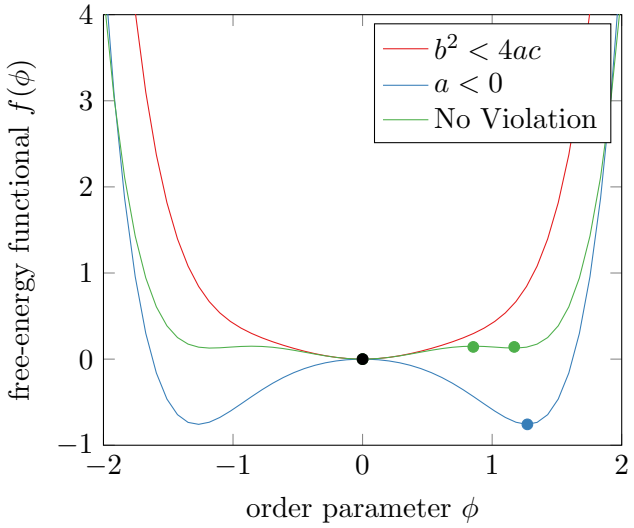


Figure 20: Illustration of the different extrema behaviors of the Landau free-energy functional in Eq. (1). If $b^2 < 4ac$, all extrema vanish, while for $a < 0$ (and $b^2 > 4ac$) only the ϕ_-^2 term vanishes. If both conditions are respected, one minima and one maxima apart from the minimum at $\phi = 0$ emerge. Values are $a = 1$, $b = -1$, $c = 1$ (red), and $a = -1$, $b = -1$, $c = 1$ (blue), and $a = 1$, $b = -2.1$, $c = 1$ (green).

ated as

$$0 = f(\phi_+^2, a_c) = \frac{a_c}{4c}(-b + \sqrt{b^2 - 4a_c c}) + \frac{b}{16c^2}(-b + \sqrt{b^2 - 4a_c c})^2 + \frac{1}{48c^2}(-b + \sqrt{b^2 - 4a_c c})^3,$$

which after some algebra yields

$$a_c = \frac{3b^2}{16c}.$$

Exactly at the critical value, the equilibrium order parameter ϕ_{eq} is not defined, but is instead degenerate between the values $\phi_{eq} = 0$ and

$$\phi_{eq}^2 = \phi_+^2(a_c) = -\frac{3b}{4c}$$

(c) Sketch the free energy $f(\phi_{eq})$ as a function of the parameter a in the vicinity of the phase transition. Why is it a first-order phase transition?
Hint: Expand $f(\phi_{eq})$ up to first order in $\delta a = a - a_c$ around $\delta a = 0$.

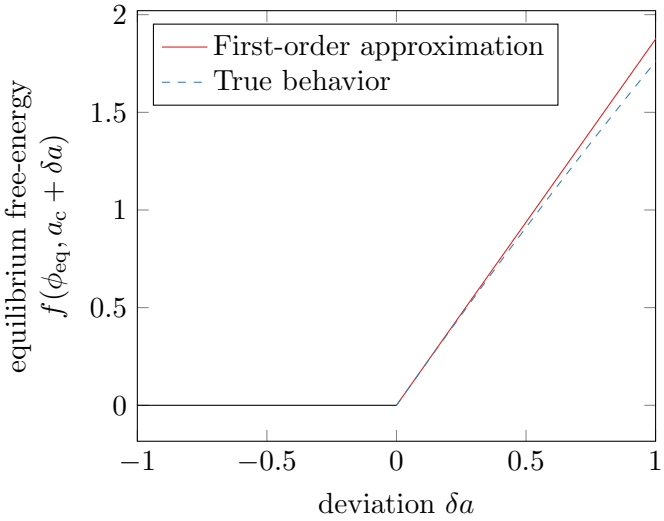


Figure 21: Illustration of the equilibrium free-energy close to the critical value a_{c} for $b = -5$ and $c = 1$.

This is a first-order phase transition because the difference between the order parameters before and after the phase transition does not approach zero, i.e. the parameter itself is discontinuous.
We expand $f(\phi_{\mathrm{eq}})$ for $\phi_{\mathrm{eq}}^2 = \phi_+^2(a_{\mathrm{c}})$ around $\delta a = a - a_{\mathrm{c}}$ up to first order $\mathcal{O}(\delta a)$. First, the equilibrium order parameter ϕ_{eq} is

$$\phi_{\mathrm{eq}}(a_{\mathrm{c}} + \delta a)^2 = \frac{-b + \sqrt{b^2 - 4ca_{\mathrm{c}} - 4c\delta a}}{2c} = -\frac{3b}{4c} + \frac{2}{b}\delta a + \mathcal{O}(\delta a^2).$$

This allows approximating the free-energy density as

$$\begin{aligned} f(\phi_{\mathrm{eq}}(a_{\mathrm{c}} + \delta a)) &= \frac{a_{\mathrm{c}} + \delta a}{2}\left(-\frac{3b}{4c} + \frac{2}{b}\delta a\right) + \frac{b}{4}\left(-\frac{3b}{4c} + \frac{2}{b}\delta a\right)^2 \\ &\quad + \frac{c}{6}\left(-\frac{3b}{4c} + \frac{2}{b}\delta a\right)^3 + \mathcal{O}(\delta a^2) \\ &= -\frac{3b}{8c}\delta a. \end{aligned}$$

For $a < a_{\mathrm{c}}$, i.e. $\delta a < 0$, the equilibrium order parameter is $\phi_{\mathrm{eq}} = 0$ and hence $f(\phi_{\mathrm{eq}}) = 0$. The general behavior is sketched in Fig. 21.

10. Phase diagram of a two-order-parameter system

Consider a system with two real order parameters ϕ_1 and ϕ_2, whose free-energy density is given by

$$f(\phi_1, \phi_2) = \frac{r}{2}(\phi_1^2 + \phi_2^2) - \frac{g}{2}(\phi_1^2 - \phi_2^2) + \frac{u}{4}(\phi_1^4 + \phi_2^4) + \frac{v}{2}\phi_1^2\phi_2^2, \tag{2}$$

where $u, v > 0$.

(a) Determine all extrema of the functional $f(\phi_1, \phi_2)$ in Eq. (2). Which values are taken by ϕ_1^2 and ϕ_2^2 at these extrema?

The function $f(\phi_1, \phi_2)$ is extremal if

$$0 = \nabla f(\phi_1, \phi_2) = \begin{pmatrix} \phi_1(r - g + u\phi_1^2 + v\phi_2^2) \\ \phi_2(r + g + u\phi_2^2 + v\phi_1^2) \end{pmatrix}.$$

Therefore, possible pairs (ϕ_1^2, ϕ_2^2) of solutions are

	ϕ_1^2	ϕ_2^2
(I)	0	0
(II)	0	$-\frac{r+g}{u}$
(III)	$-\frac{r-g}{u}$	0
(IV)	$-\frac{r}{u+v} + \frac{g}{u-v}$	$-\frac{r}{u+v} - \frac{g}{u-v}$

These are the values taken by (ϕ_1^2, ϕ_2^2) at the extrema of the free-energy functional.

(b) Which conditions have to be posed on ϕ_1^2 and ϕ_2^2? Discuss which phases (i.e., configurations of ϕ_1 and ϕ_2) are physically reasonable in which areas of the (r, g) plane.

In order for the solutions to be physically reasonable, they have to be positive, hence

$$\phi_1^2 \geq 0, \qquad \phi_2^2 \geq 0,$$

has to be imposed. This restricts the respective extrema to the following domains:

(I) No restrictions.

(II) From $-\frac{r+g}{u} \geq 0$ it follows that $r + g \leq 0$ and hence

$$r \leq -g.$$

(III) From $-\frac{r-g}{u} \geq 0$ it follows that $r - g \leq 0$ and hence

$$r \leq g.$$

(IV) The respective bounds need to be computed for $u^2 > v^2$ and $u^2 < v^2$ separately. Starting with $u^2 > v^2$, from $-\frac{r}{u+v} + \frac{g}{u-v} \geq 0$ it follows that

$$r \leq \frac{u+v}{u-v} g,$$

while from $-\frac{r}{u+v} - \frac{g}{u-v} \geq 0$ it follows that

$$r \leq -\frac{u+v}{u-v} g.$$

Here, the second inequality is more restrictive for $g > 0$, while the first is more restrictive for $g < 0$.

If, however, $u^2 < v^2$, the respective bounds are

$$r \leq -\frac{u+v}{v-u} g, \qquad\qquad r \leq \frac{u+v}{v-u} g.$$

In summary, we find

$$r \leq -\frac{u+v}{u-v} |g| \quad \text{for} \quad u^2 > v^2, \qquad r \leq -\frac{u+v}{v-u} |g| \quad \text{for} \quad u^2 < v^2.$$

(c) In each case, determine the state with the lowest free energy as function of r and g. Distinguish between $u^2 < v^2$ and $u^2 > v^2$.

The free energies at the respective solutions are

$$\begin{aligned}
f_{\mathrm{I}}(r,g) &= f(0,0) = 0, \\
f_{\mathrm{II}}(r,g) &= f(0, -\frac{r+g}{u}) = -\frac{(r+g)^2}{4u}, \\
f_{\mathrm{III}}(r,g) &= f(-\frac{r-g}{u}, 0) = -\frac{(r-g)^2}{4u}, \\
f_{\mathrm{IV}}(r,g) &= f(-\frac{r}{u+v} + \frac{g}{u-v}, -\frac{r}{u+v} - \frac{g}{u-v}) = -\frac{1}{2}\left(\frac{r^2}{u+v} + \frac{g^2}{u-v}\right).
\end{aligned}$$

In particular, we have at the boundaries of (II) and (III) derived in (b) that $f_{\mathrm{II}}(r,-g) = 0$ and $f_{\mathrm{III}}(r,g) = 0$. In their respective domains, we have $f_{\mathrm{II}} < 0$ and $f_{\mathrm{III}} < 0$, and hence especially $f_{\mathrm{II}}, f_{\mathrm{III}} < f_{\mathrm{I}}$. The two extrema are equal for

$$f_{\mathrm{II}}(r,g) = -\frac{(r+g)^2}{4u} = f_{\mathrm{III}}(r,g) = -\frac{(r-g)^2}{4u} \implies g = 0,$$

i.e. along the g-axis. The respective extrema (II) and (III) are lower than the other in the following domains (irrespective of the restrictions posed in the previous task):

	$r<0$	$r>0$
$g<0$	II	III
$g>0$	III	II

To incorporate (IV), we check

$$f_{\rm IV}(r,g) < f_{\rm II}(r,g) \implies 0 < r^2\frac{u-v}{u+v} + g^2\frac{u+v}{u-v} - 2rg,$$
$$f_{\rm IV}(r,g) < f_{\rm III}(r,g) \implies 0 < r^2\frac{u-v}{u+v} + g^2\frac{u+v}{u-v} + 2rg.$$

In the case of $u^2 > v^2$, we have (for the solution of the above inequalities valid in the domains in which (IV) is physically feasible)

$$f_{\rm IV}(r,g) < f_{\rm II}(r,g) \implies r < \frac{u+v}{u-v}g,$$
$$f_{\rm IV}(r,g) < f_{\rm III}(r,g) \implies r < -\frac{u+v}{u-v}g.$$

However, if $u^2 < v^2$, we neither find $f_{\rm IV}(r,g) < f_{\rm II}(r,g)$ nor $f_{\rm IV}(r,g) < f_{\rm III}(r,g)$. Hence, the phase (IV) does not materialize.

(d) Sketch the phase diagram in the (r,g) plane for $u^2 < v^2$ and $u^2 > v^2$, respectively. What are the orders of the different phase transitions?

The phase diagram is sketched in Fig. 22.
The transitions from (I) to (II) or (III) are second order, because the order parameter ϕ_1^2 and ϕ_2^2 change continuously; the transition from (II) to (III) at $g = 0$ is first-order, because the order parameter (ϕ_1^2, ϕ_2^2) change discontinuously from $(0, -r/u)$ to $(-r/u, 0)$. Transitioning from (IV) to (II) or (III) is second-order, as again the pair of order parameters change continuously.

11. Tricritical point in an antiferromagnet

An external magnetic field h applied to an antiferromagnet couples to the total magnetization m instead of the antiferromagnetic order parameter, the staggered magnetization n. Assume that the coupling between m and n is described phenomenologically by the Landau free energy density

$$f(n,m) = \frac{t}{2}n^2 + \frac{b}{4}n^4 + \frac{v}{2}m^2 + \frac{w}{2}n^2m^2 - hm, \tag{3}$$

where $t = (T - T_0)a$, and a, b, v, w are positive constants.

(a) Show that this model features a paramagnetic phase with magnetization m_0. Derive a temperature-independent relation $m = m(n^2)$ between the magnetization m

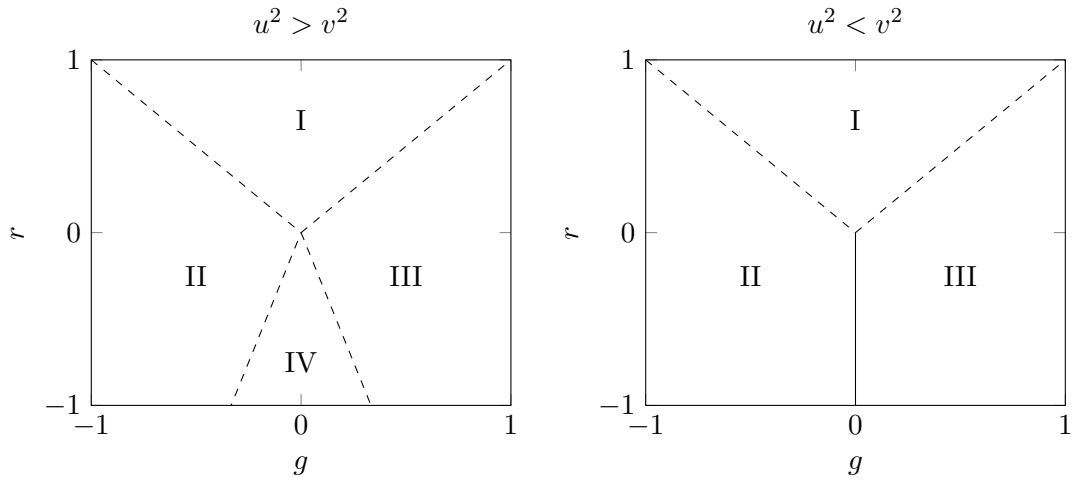


Figure 22: Phase diagrams for the free-energy density in Eq. (2) for $u^2 > v^2$ ($u = 2$, $v = 1$ in the sketch) and $u^2 < v^2$ ($u = 1$, $v = 2$ in the sketch). First-order transitions are depicted solid, while second order transitions are dashed.

and the staggered magnetization n in the antiferromagnetic phase, i.e., for $n^2 > 0$.
Hint: Equilibrium states are given by minima of $f(n, m)$.

We find the equilibrium states, by considering

$$0 = \nabla f(n, m) = \begin{pmatrix} (t + bn^2 + wm^2)n \\ (v + wn^2)m - h \end{pmatrix}$$

From the second component, it follows that for $v + wn^2 \neq 0$, that

$$m(n^2) = \frac{h}{v + wn^2}.$$

The first component then admits the solutions $n = 0$ and

$$bw^2n^6 + (tw^2 + 2vwb)n^4 + (2tvw + bv^2)n^2 + (tv^2 + wh^2) = 0.$$

Here, the solution $n = 0$ and $m = h/v \equiv m_0$ corresponds to the paramagnetic phase, which we check to be stable by considering the second derivative

$$\nabla^2 f(n = 0, m = h/v) = \begin{pmatrix} t + 3bn + wm^2 & 2wmn \\ 2wmn & v + wn^2 \end{pmatrix}\Bigg|_{\substack{n=0 \\ m=h/v}} = \begin{pmatrix} t + \frac{wh^2}{v^2} & 0 \\ 0 & v \end{pmatrix} > 0,$$

which is indeed positive. Hence, this is a minimum and at least a metastable paramagnetic phase.

(b) Consider the antiferromagnetic phase near the phase transition, i.e., for small

values of n^2. Write $m = m_0 + \delta m$, expand $m(n^2)$ for small n^2, and derive a relation between δm and n^2.

We expand

$$m(n^2) = \frac{h}{v}\frac{1}{1+\frac{w}{v}n^2} = \frac{h}{v}\left(1 - \frac{w}{v}n^2 + \mathcal{O}(n^4)\right) = m_0 - \underbrace{\frac{hw}{v^2}n^2}_{\delta m} + \mathcal{O}(n^4),$$

finding

$$\delta m = -\frac{hw}{v^2}n^2.$$

(c) Show that the effective free energy density for the staggered magnetization

$$g(n) = f(n, m_0 + \delta m) - f(0, m_0)$$

can be written as

$$g(n) = \frac{\overline{a}}{2}n^2 + \frac{\overline{b}}{4}n^4 + \frac{\overline{c}}{6}n^6 + \mathcal{O}(n^8), \tag{4}$$

with to-be-determined temperature- and field-dependent coefficients $\overline{a}$, $\overline{b}$ and $\overline{c}$.

The two terms are

$$\begin{aligned} f(0, m_0) &= \frac{v}{2}m_0^2 - hm_0, \\ f(n, m_0 + \delta m) &= \frac{t}{2}n^2 + \frac{b}{4}n^4 + \left(\frac{v}{2} + \frac{w}{2}n^2\right)(m_0^2 + 2m_0\delta m + \delta m^2) - hm_0 - h\delta m, \end{aligned}$$

such that the effective free energy density is

$$\begin{aligned} g(n) &= \frac{t}{2}n^2 + \frac{b}{4}n^4 + \frac{v}{2}(2m_0\delta m + \delta m^2) + \frac{w}{2}n^2(m_0^2 + 2m_0\delta m + \delta m^2) - h\delta m \\ &= \underbrace{\left(\frac{t}{2} + \frac{h^2 w}{2v^2}\right)}_{\frac{\overline{a}}{2}} n^2 + \underbrace{\left(\frac{b}{4} - \frac{w^2h^2}{2v^3}\right)}_{\frac{\overline{b}}{4}} n^4 + \underbrace{\frac{w^3h^2}{2v^4}}_{\frac{\overline{c}}{6}} n^6, \end{aligned}$$

yielding the to-be-determined coefficients

$$\overline{a} = t + \frac{h^2 w}{v^2}, \qquad \overline{b} = b - 2\frac{w^2h^2}{v^3}, \qquad \overline{c} = 3\frac{w^3h^2}{v^4}.$$

(d) Show that the model in Eq. (3) features a tricritical point at temperature T_t and field h_t, where

$$T_\mathrm{t} = T_0 - \frac{bv}{2aw}, \quad h_\mathrm{t}^2 = \frac{bv^3}{2w^2}. \tag{5}$$

Hint: Here we define a tricritical point as a point where first- and second-order transition lines meet. Depending on the sign of the coefficient $\overline{b}$ in $g(n)$, the transition between the paramagnetic and antiferromagnetic phases are first or second order, cf. Problem 1 on Exercise 1.

The values of our previously determined coefficients at the tricritical point are

$$\overline{a}_{\mathrm{tc}} = a(T_{\mathrm{t}} - T_0) + \frac{h_{\mathrm{t}}^2 w}{v^2} = 0, \quad \overline{b}_{\mathrm{tc}} = b - 2\frac{w^2 h_{\mathrm{t}}^2}{v^3} = 0, \quad \overline{c}_{\mathrm{tc}} = 3\frac{w^3 h_{\mathrm{t}}^2}{v^2} = \frac{3bv}{2w}.$$

In Problem 1 on Exercise 1, we found that the effective free energy in Eq. (4) exhibits a first-order phase transition for $\overline{b} < 0$ and $\overline{c} > 0$ at

$$\overline{a}_{\mathrm{c}} = \frac{3\overline{b}^2}{16\overline{c}},$$

which for $\overline{b} = 0$ yields the value $\overline{a}_{\mathrm{c}} = 0 = \overline{a}_{\mathrm{tc}}$ at the tricritical point. Hence, this is a point at which lies on the first-order phase transition line.
For $\overline{b} > 0$, the phase transitions are second order, because the order parameter smoothly changes from $n^2 = 0$ ($\overline{a} < \overline{a}_{\mathrm{a}}$) to $n^2 \neq 0$ ($\overline{a} > \overline{a}_{\mathrm{a}}$).

(e) Show that the second-order phase transition occurs for $h < h_{\mathrm{t}}$ at

$$T_{\mathrm{c}} = T_0 - \frac{wh^2}{av^2}, \tag{6}$$

and the first-order transition occurs for $h > h_{\mathrm{t}}$ at

$$T_{\mathrm{c}} = T_0 - \frac{3wh^2}{4av^2} - \frac{bv}{4aw} + \frac{b^2v^4}{16aw^3h^2}. \tag{7}$$

To show Eq. (7), we use the criterion derived in Problem 1 on Exercise 1 for the first-order transition, namely

$$\overline{b} = b - 2\frac{w^2h^2}{v^3} \overset{!}{<} 0 \implies h^2 > \frac{bv^3}{2w^2} = h_{\mathrm{t}}^2,$$

i.e. the first-order phase transition occurs for $h > h_{\mathrm{t}}$. Furthermore, from the derived condition on the critical value $\overline{a}_{\mathrm{c}}$, we obtain

$$\overline{a}_{\mathrm{c}} = a(T_{\mathrm{c}} - T_0) + \frac{h^2 w}{v^2} = \frac{3\overline{b}^2}{16\overline{c}} = \frac{1}{16}\left(\frac{b^2v^4}{w^3h^2} - 4\frac{bv}{w} + 4\frac{wh^2}{v^2}\right),$$

such that

$$T_{\mathrm{c}} = T_0 - \frac{3wh^2}{4av^2} - \frac{bv}{4aw} + \frac{b^2v^4}{16aw^3h^2}.$$

Turning to Eq. (6) describing the temperature at which the second-order phase

transition occurs, we notice that the minima of Eq. (4) are given by

$$n_0 = 0, \qquad n_\pm^2 = \frac{-\bar{b} \pm \sqrt{\bar{b}^2 - 4\bar{a}\bar{c}}}{2\bar{c}}$$

If $\bar{b} > 0$ and hence $h < h_\mathrm{t}$, the $n_\pm^2$ are always negative or imaginary, i.e. invalid solutions, for $\bar{a} > 0$, while n_+^2 always exists for $\bar{a} < 0$. At $\bar{a} = 0$, we have $n_+ = 0 = n_0$ making this transition second order. It occurs at

$$\bar{a}_\mathrm{c} = 0 = a(T_\mathrm{c} - T_0) + \frac{wh^2}{v^2} \implies T_\mathrm{c} = T_0 - \frac{wh^2}{av^2}.$$

12. Static scaling hypothesis

Consider the static scaling hypothesis for the free energy density

$$f_\mathrm{s}(t, h) = b^{-d} f_\mathrm{s}(b^{y_t} t, b^{y_h} h) \tag{8}$$

with scaling exponents y_t for the reduced temperature t and y_h for the external field h.

(a) Use the static scaling hypothesis to derive the relation

$$\delta = \frac{d + 2 - \eta}{d - 2 + \eta} \tag{9}$$

between the critical-isotherm exponent δ and the anomalous dimension η.
Hint: Use the relation $y_t = 1/\nu$ and Fisher's law $\gamma = \nu(2 - \eta)$ derived in class.

We seek the scaling exponent δ relating the external field h and the order parameter ϕ as

$$h \propto \phi^\delta.$$

The Ginzburg-Landau functional is

$$f(t, h, \phi) = f_n + f_0 \left[\frac{a}{2}\phi^2 + \frac{b}{4}\phi^4 + \xi_0^2 (\nabla\phi)^2 - \phi h \right] + \mathcal{O}(\phi^6, \nabla^4, \nabla^2\phi^4),$$

hence

$$\phi = \frac{\partial f}{\partial h} + \mathcal{O}(h).$$

From this, we obtain the scaling behavior of the order parameter ϕ as

$$\phi(t, h) = \frac{\partial f_\mathrm{s}(t, h)}{\partial h} \xrightarrow[t \mapsto b^{y_t} t]{h \mapsto b^{y_h} h} b^{-d} \frac{\partial f_\mathrm{s}(b^{y_t} t, b^{y_h} h)}{\partial (b^{y_h} h)} \frac{\partial (b^{y_h} h)}{\partial h} = b^{-d+y_h} \phi(b^{y_t} t, b^{y_h} h).$$

Setting $h = 0$ and $b^{-d} = t^{d\nu}$, cf. Lecture notes, we find

$$\phi(t,0) \propto t^{-\nu(-d+y_h)}\phi(b^{y_t}t,0) \overset{!}{\propto} t^{\beta}\phi(b^{y_t}t,0),$$

yielding $\beta = \nu(d - y_h)$. To repeat this same trick with the external field h, set $t = 0$ and $b^{y_h}h = 1$, such that $b^{-d} = h^{\frac{d}{y_h}}$. Then, the free energy functional scales as

$$f(0,h) = h^{\frac{d}{y_h}} f(0,1) \propto h^{\frac{d}{y_h}}.$$

Now,

$$\phi = \frac{\partial f}{\partial h} \propto \frac{\partial h^{\frac{d}{y_h}}}{\partial h} = h^{\frac{d-y_h}{y_h}} \overset{!}{\propto} h^{\frac{1}{\delta}},$$

and therefore $\delta = \frac{y_h}{d-y_h}$. By considering the inverse susceptibility, we obtain

$$\frac{1}{\chi} = \frac{\partial h}{\partial \phi} \propto \frac{\partial \phi^{\delta}}{\partial \phi} = \phi^{\delta-1} \propto t^{\beta(\delta-1)} \overset{!}{\propto} t^{\gamma},$$

i.e. $\beta(\delta - 1) = \gamma$.
We hence found

$$\beta = \nu(d - y_h), \qquad \gamma = \beta(\delta - 1), \qquad \delta = \frac{y_h}{d - y_h}.$$

By eliminating β and rewriting the first two, we obtain y_h as

$$\gamma = \nu(d - y_h)(\delta - 1) \implies y_h = d - \frac{\gamma}{\nu(\delta - 1)} = d - \frac{2 - \eta}{\delta - 1}.$$

Hence,

$$\delta = \frac{d(\delta - 1) - (2 - \eta)}{2 - \eta} \implies \delta = \frac{d + (2 - \eta)}{d - (2 - \eta)}.$$

(b) In principle, critical exponents above and below a transition could differ from each other. Show for the example of the correlation-length exponents ν and ν' above and below the transition, respectively, that the static scaling hypothesis implies that they are equal, $\nu(T > T_c) = \nu'(T < T_c)$.
Hint: For fixed $h \neq 0$, $f_s(t,h)$ should be a smooth function of t, because the only singularity that we expect is at $t = h = 0$. Show that $f_s(t,h)$ can be written in the form

$$f_s(t,h) = h^{d/y_h} F_f^{\pm}\left(\frac{|t|}{h^{1/(\overline{\nu} y_h)}}\right), \tag{10}$$

and explain how the smoothness assumption mentioned above constrains the analytic form of the function $F_f^{\pm}(x)$.

In the free energy density in Eq. (3), for fixed h, we set $b^{y_h}h = 1$ resulting in $b^{-d} = h^{\frac{d}{y_h}}$, finding from the scaling hypothesis

$$f_{\mathrm{s}}(t,h) = h^{\frac{d}{y_h}} f_{\mathrm{s}}\left(\frac{1}{h^{\frac{y_t}{y_h}}}t, 1\right) = h^{\frac{d}{y_h}} f_{\mathrm{s}}\left(\frac{t}{h^{\frac{1}{\nu y_h}}}, 1\right)$$

By defining

$$F_f^+\left(\frac{|t|}{h^{\frac{1}{\nu y_h}}}\right) \equiv f_{\mathrm{s}}\left(\frac{+|t|}{h^{\frac{1}{\nu y_h}}}, 1\right), \qquad F_f^-\left(\frac{|t|}{h^{\frac{1}{\nu y_h}}}\right) \equiv f_{\mathrm{s}}\left(\frac{-|t|}{h^{\frac{1}{\nu' y_h}}}, 1\right),$$

we obtain the form in Eq. (10). For $f_{\mathrm{s}}(t,h)$ to be smooth except at $t = h = 0$, it needs to be in C^∞, i.e. its Taylor expansion needs to exist. This means however, that also the $F_f^\pm$ need to have a Taylor expansion at the chosen point and that they have to be equal on their shared domain. The first order term ensures that y_h is the same at $T < T_{\mathrm{c}}$ and $T > T_{\mathrm{c}}$, while $h^{-1/\nu y_h}$ and $h^{-1/\nu' y_h}$ entering the higher-order coefficients ensures that also $\nu = \nu'$.

13. Generating functional for noninteracting real bosons

Consider the (discretized) field theory of a noninteracting real scalar boson field $\phi \equiv (\phi_k)_{k=1}^M$ with action

$$S[\phi] = \sum_{k,l=1}^{M} \frac{1}{2}\phi_k K_{kl} \phi_l, \tag{11}$$

and positive definite symmetric and real matrix $K = K^\intercal$ (kernel).

(a) Show that the n-point correlation functions

$$\langle \phi_{l_1} \dots \phi_{l_n} \rangle \equiv \frac{1}{Z[0]} \int \prod_{k=1}^{M} \frac{\mathrm{d}\phi_k}{\sqrt{2\pi}} \phi_{l_1} \cdots \phi_{l_n} \exp(-S[\phi]) \tag{12}$$

can be obtained from the *generating functional*

$$Z[h] = \int \prod_{k=1}^{M} \frac{\mathrm{d}\phi_k}{\sqrt{2\pi}} \exp\left(-S[\phi] + \sum_{k=1}^{M} h_k \phi_k\right) \tag{13}$$

via suitable derivatives with respect to the external source ("magnetic field") h.

The respective translation is

$$\langle \phi_{l_1} \dots \phi_{l_n} \rangle = \frac{1}{Z[0]}\, \partial_{l_1} \cdots \partial_{l_n} Z[h]|_{h=0}\,.$$

(b) Show that the generating functional for a noninteracting real scalar boson field theory can be computed in closed form as

$$Z[h] = (\det K)^{-1/2} \exp\left(\sum_{k,l=1}^{M} \frac{1}{2} h_k (K^{-1})_{kl} h_l\right) \tag{14}$$

From now on, we use the Einstein summation convention. The real and symmetric matrix $K = K^\intercal$ is decomposable as $K = Q\Lambda Q^\intercal$ with Q orthogonal, i.e. $QQ^\intercal = I$.

$$\begin{aligned} Z[h] &= \int \prod \frac{\mathrm{d}\phi_k}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\phi_k K_{kl} \phi_l + h_k \phi_k\right) \\ &= \int \prod \frac{\mathrm{d}\phi_k}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\phi_k Q_{km}) \lambda_m ((Q^\intercal)_{ml} \phi_l) + h_k \phi_k\right) \end{aligned}$$

Now, we substitute $\psi_k = (Q^\intercal)_{kl}\phi_l$ and $\psi_k = \phi_l Q_{lk}$, and $\mathrm{d}\psi_k = \det Q \mathrm{d}\phi_k = \mathrm{d}\phi_k$ as $\det Q = 1$. Then,

$$\begin{aligned} Z[h] &= \int \prod \frac{\mathrm{d}\psi_k}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\psi_m \lambda_m \psi_m + h_k Q_{km} \psi_m\right) \\ &= \int \prod \frac{\mathrm{d}\psi_k}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\lambda_m \psi_m^2 + \tilde{h}_m \psi_m\right) \\ &= \exp\left(\frac{\tilde{h}_m^2}{2\lambda_m}\right) \int \prod \frac{\mathrm{d}\psi_k}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\sqrt{\lambda_m}\psi_m - \frac{\tilde{h}_m}{\sqrt{\lambda_m}}\right)^2\right) \end{aligned}$$

with $\tilde{h}_m = (Q^\intercal)_{mk} h_k$. Now, let $\tilde{\psi}_k = \sqrt{\lambda_k}\psi_k - \frac{\tilde{h}_m}{\sqrt{\lambda_m}}$ and $\lambda_k^{-1/2}\mathrm{d}\tilde{\psi}_k = \mathrm{d}\psi_k$, yielding

$$\begin{aligned} Z[h] &= \lambda_k^{-1/2} \exp\left(\frac{\tilde{h}_m^2}{2\lambda_m}\right) \int \prod \frac{\mathrm{d}\tilde{\psi}_k}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\tilde{\psi}_k^2\right) \\ &= (\det K)^{-1/2} \exp\left(\frac{1}{2} h_k Q_{km} \lambda_m^{-1} (Q^\intercal)_{ml} h_l\right) \\ &= (\det K)^{-1/2} \exp\left(\frac{1}{2} h_k (K^{-1})_{kl} h_l\right). \end{aligned}$$

(c) Use the above result to show that the propagator $G^{(2)}_{kl} \equiv \langle \phi_k \phi_l \rangle$ and the four-point function $G^{(4)}_{klmn} = \langle \phi_k \phi_l \phi_m \phi_n \rangle$ can be written as

$$G^{(2)}_{kl} = (K^{-1})_{kl} \qquad \text{and} \qquad G^{(4)}_{klmn} = G^{(2)}_{kl} G^{(2)}_{mn} + G^{(2)}_{km} G^{(2)}_{ln} + G^{(2)}_{kn} G^{(2)}_{lm}. \tag{15}$$

For the propagator $G^{(2)}_{kl}$, we find

$$\begin{aligned}
G^{(2)}_{kl} &= \langle \phi_k \phi_l \rangle = \frac{1}{Z[0]} \partial_k \partial_l Z[h]|_{h=0} \\
&= \frac{1}{(\det K)^{-1/2}} \, \partial_k \partial_l (\det K)^{-1/2} \exp\left(\frac{1}{2} h_m (K^{-1})_{mn} h_n \right) \Big|_{h=0} \\
&= \frac{1}{2} \, \partial_k \exp\left(\frac{1}{2} h_m (K^{-1})_{mn} h_n \right) \left((K^{-1})_{ln} h_n + h_m (K^{-1})_{ml} \right) \Big|_{h=0} \\
&= \frac{1}{2} \exp\left(\frac{1}{2} h_m (K^{-1})_{mn} h_n \right) \\
&\qquad \left(\frac{1}{2} \left((K^{-1})_{ln} h_n + h_m (K^{-1})_{ml} \right) + (K^{-1})_{lk} + (K^{-1})_{kl} \right) \Big|_{h=0} \\
&= \frac{1}{2} ((K^{-1})_{lk} + (K^{-1})_{kl})
\end{aligned}$$

If K is symmetric, then K^{-1} is symmetric as well and hence

$$G^{(2)}_{kl} = (K^{-1})_{kl}.$$

Now, for the four-point correlation function $G^{(4)}_{klmn} = \langle \phi_k \phi_l \phi_m \phi_n \rangle$, we find

$$G^{(4)}_{klmn} = \frac{1}{Z[0]} \partial_k \partial_l \partial_m \partial_n Z[h]|_{h=0}$$

14. Partition function for complex bosons

Use the result of Problem 1 to show that the partition function $Z \equiv Z[0]$ for the theory of noninteracting complex boson fields Φ, Φ^* is

$$Z = \int \prod_{k=1}^{M} \frac{\mathrm{d}\phi_k^* \mathrm{d}\phi_k}{2\pi i} \exp\left(- \sum_{k,l=1}^{M} \phi_k^* K_{kl} \phi_l \right) = (\det K)^{-1}. \tag{16}$$

assuming a positive definite Hermitian kernel $K = K^\dagger$.

If $K = K^\dagger$, then $K = U \Lambda U^\dagger$. Hence,

$$\begin{aligned}
Z &= \int \prod \frac{\mathrm{d}\phi_k^* \mathrm{d}\phi_k}{2\pi i} \exp(-\phi_k^* K_{kl} \phi_l) \\
&= \int \prod \frac{\mathrm{d}\phi_k^* \mathrm{d}\phi_k}{2\pi i} \exp(-(\phi_k^\dagger U_{km}) \lambda_m (U^\dagger_{ml} \phi_l))
\end{aligned}$$

Now, we consider the decomposition into real and imaginary part ϕ_k^r and ϕ_k^i of ϕ_k. This is given by the transformation

$$\begin{pmatrix} \phi_k^r \\ \phi_k^i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \phi_k \\ \phi_k^* \end{pmatrix} = A \begin{pmatrix} \phi_k \\ \phi_k^* \end{pmatrix}$$

with $\det A = \frac{2i}{4} = \frac{i}{2}$, therefore $\det A^{-1} = -2i$. Hence, using $\tilde{\phi}_k^{r/i} = U_{kl}^\dagger \phi_l^{r/i}$, the integral transforms into

$$\begin{aligned} Z &= \int \prod \frac{\mathrm{d}\phi_k^r \mathrm{d}\phi_k^i}{\pi} \exp\left(-((\phi_k^r - i\phi_k^i)U_{km})\lambda_m (U_{ml}^\dagger(\phi_l^r + i\phi_l^i))\right) \\ &= \int \prod \frac{\mathrm{d}\tilde{\phi}_k^r \mathrm{d}\tilde{\phi}_k^i}{\pi} \exp\left(-\lambda_m \left((\tilde{\phi}_m^r)^2 + (\tilde{\phi}_m^i)^2\right)\right) \\ &= \frac{1}{\lambda_m} = (\det K)^{-1} \end{aligned}$$

15. Susceptibility exponent γ in the large-N limit

Consider the partition function for the theory of N complex boson fields Φ_a and Φ_a^*, $a = 1, \ldots, N$, interacting via an ultralocal two-body interaction,

$$Z = \int \prod_{a=1}^{N} \mathcal{D}\Phi_a^*(\vec{x}) \mathcal{D}\Phi_a(\vec{x}) e^{-S[\Phi^*, \Phi]} \tag{17}$$

with action

$$S[\Phi^*, \Phi] = \int \mathrm{d}^d \vec{x} \left[\sum_{a=1}^{N} (|\nabla \Phi_a(\vec{x})|^2 + t|\Phi_a(\vec{x})|^2) + \frac{\lambda}{2N} \left(\sum_{a=1}^{N} |\Phi_a(\vec{x})|^2 \right)^2 \right]. \tag{18}$$

Here, t is the tuning parameter for a classical phase transition distinguishing the disordered phase for $t > t_c$ from an ordered phase for $t < t_c$ and λ denotes the quartic coupling.

(a) Show that the partition function can be written as

$$Z = \int \prod_{a=1}^{N} \mathcal{D}\Phi_a^*(\vec{x}) \mathcal{D}\Phi_a(\vec{x}) \mathcal{D}\sigma(\vec{x}) e^{-S_0[\Phi^*, \Phi] - \int \mathrm{d}^d \vec{x} \left[\frac{N}{2\lambda} \sigma^2(\vec{x}) + i\sigma(\vec{x}) |\Phi(\vec{x})|^2 \right]} \tag{19}$$

where we have introduced the composite field $\sigma(\vec{x})$ which couples to $|\Phi(\vec{x})|^2 = \sum_{a=1}^{N} |\Phi_a(\vec{x})|^2$ and S_0 denotes the Gaussian part of the action S. (This is the so-called *Hubbard-Stratonovich* transformation.)

The Gaussian part S_0 of the action S is

$$S_0[\Phi^*, \Phi] = |\nabla \Phi_a(\vec{x})|^2 + t\,|\Phi_a(\vec{x})|^2\,.$$

The equivalence of Eq. (17) and Eq. (19) can be shown by explicitly performing the Gaussian integral with respect to σ, by first rewriting

$$Z = \int \prod_a \mathcal{D}\Phi_a^*(\vec{x})\mathcal{D}\Phi_a(\vec{x}) e^{-S_0[\Phi^*,\Phi]} \int \mathcal{D}\sigma(\vec{x}) e^{-\int \mathrm{d}^d\vec{x}\left[\frac{N}{2\lambda}\sigma^2(\vec{x}) + i\sigma(\vec{x})|\Phi(\vec{x})|^2\right]}.$$

The exponent can be used to complete the square as

$$\left[\frac{N}{2\lambda}\sigma^2(\vec{x}) + i\sigma(\vec{x})|\Phi(\vec{x})|^2\right] = \left(\sqrt{\frac{N}{2\lambda}}\sigma + \frac{i}{2}\sqrt{\frac{2\lambda}{N}}\,|\Phi|^2\right) + \frac{\lambda}{2N}(|\Phi|^2)^2,$$

hence

$$\begin{aligned} Z &= \int \prod_a \mathcal{D}\Phi_a^*(\vec{x})\mathcal{D}\Phi_a(\vec{x}) e^{-S_0[\Phi^*,\Phi] - \frac{\lambda}{2N}(|\Phi|^2)^2} \int \mathcal{D}\sigma(\vec{x}) e^{-\int \mathrm{d}^d\vec{x}\left[\sqrt{\frac{N}{2\lambda}}\sigma + \frac{i}{2}\sqrt{\frac{2\lambda}{N}}|\Phi|^2\right]^2} \\ &= \sqrt{\frac{2\lambda\pi}{N}} \int \prod_{a=1}^{N} \mathcal{D}\Phi_a^*(\vec{x})\mathcal{D}\Phi_a(\vec{x}) e^{-S[\Phi^*,\Phi]}. \end{aligned}$$

The partition function does not care about the constant prefactor, hence Eq. (17) and Eq. (19) describe the same physics.

(b) Integrate over all components Φ_a, $2 \le a \le N$, except the first one, to obtain an effective theory in Φ_1 and σ. Consider the limit $N \to \infty$, argue that the saddle-point approximation discussed in class becomes exact in this limit, and use it to compute the free energy density.
Hint: The resulting free energy density reads

$$\frac{f}{N k_B T} = (t+\sigma)|\Phi_1|^2 - \frac{\sigma^2}{2\lambda} + \frac{1}{V}\int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \ln(k^2 + t + \sigma),$$

with the saddle-point conditions

$$(t+\sigma)\Phi_1 = 0 \qquad \text{and} \qquad \sigma = \lambda \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{k^2 + t + \sigma} + \lambda|\Phi_1|^2$$

where $V \equiv \int \mathrm{d}^d\vec{x}$ is the spatial volume, and we have assumed uniform fields $|\Phi_1|^2 \equiv \frac{1}{V}\int \mathrm{d}^d\vec{x}|\Phi_1(\vec{x})|^2$ and $\sigma \equiv \frac{1}{V}\int \mathrm{d}^d\vec{x}\sigma(\vec{x})$ at the saddle point, rotated $i\sigma \mapsto \sigma$, and rescaled $\Phi_1/\sqrt{N} \mapsto \Phi_1$.

We factor the partition function

$$\begin{aligned} Z &= \int \prod_a \mathcal{D}\Phi_a^*(\vec{x})\mathcal{D}\Phi_a(\vec{x})\mathcal{D}\sigma(\vec{x}) e^{-\int \mathrm{d}^d\vec{x}\left[|\nabla\Phi_a(\vec{x})|^2 + (t+i\sigma(\vec{x}))|\Phi_a(\vec{x})|^2 + \frac{N}{2\lambda}\sigma^2(\vec{x})\right]} \\ &= \int \mathcal{D}\sigma(\vec{x}) e^{-\int \mathrm{d}^d\vec{x}\frac{N}{2\lambda}\sigma^2(\vec{x})} \prod_a I_a(\sigma) \end{aligned}$$

with

$$I_a(\sigma) = \int \mathcal{D}\Phi_a^*(\vec{x})\mathcal{D}\Phi_a(\vec{x}) e^{-\int \mathrm{d}^d\vec{x}\left[|\nabla\Phi_a(\vec{x})|^2 + (t+i\sigma(\vec{x}))|\Phi_a(\vec{x})|^2\right]}$$
$$\equiv \int \mathcal{D}\Phi_a^*(\vec{x})\mathcal{D}\Phi_a(\vec{x}) e^{-\int \mathrm{d}^d\vec{x} S_a(\Phi_a^*,\Phi_a,\sigma)}.$$

The action S_a may be rewritten (using the assumption, that the fields vanish as $|x| \to \infty$) into

$$-\int \mathrm{d}^d\vec{x} S_a(\Phi_a^*,\Phi_a,\sigma) = -\int \mathrm{d}^d\vec{x}\left[(\nabla\Phi_a^*)(\nabla\Phi_a) + (t+i\sigma)\Phi_a^*\Phi_a\right]$$
$$= -\int \mathrm{d}^d\vec{x}\left[\underbrace{\nabla(\Phi_a^*(\nabla\Phi_a))}_{\text{vanishes}} - \Phi_a^*\nabla^2\Phi_a + (t+i\sigma)\Phi_a^*\Phi_a\right]$$
$$= -\int \mathrm{d}^d\vec{x}\left[\Phi_a^*(-\nabla^2 + (t+i\sigma))\Phi_a\right] \equiv -\Phi_a^* K_\sigma \Phi_a$$

Using the result from Exercise 2, we find (up to a prefactor, which is irrelevant for Z), that

$$I_a(\sigma) = (\det K_\sigma)^{-1},$$

and hence

$$Z = \int \mathcal{D}\sigma e^{-\int \mathrm{d}^d\vec{x}\frac{N}{2\lambda}\sigma^2} \int \mathcal{D}\Phi_1^*\mathcal{D}\Phi_1 e^{-\int \mathrm{d}^d\vec{x} S_1(\Phi_1^*,\Phi_1,\sigma)} (\det K_\sigma)^{-(N-1)}$$
$$= \int \mathcal{D}\sigma\mathcal{D}\Phi_1^*\mathcal{D}\Phi_1 e^{-\int \mathrm{d}^d\vec{x} S_1(\Phi_1^*,\Phi_1,\sigma)} \det(K_\sigma) \left(\frac{e^{-\int \mathrm{d}^d\vec{x}\frac{1}{2\lambda}\sigma^2}}{\det K_\sigma}\right)^N,$$

which is an effective field theory for Φ_1 and σ. In order to obtain the saddle point, let $\sigma \equiv \frac{1}{V}\int \mathrm{d}^d\vec{x}\sigma(\vec{x})$ be uniform and independent of $\vec{x}$. Then, consider the Fourier transform $\Phi_a(\vec{x}) = \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\tilde{\Phi}_a(\vec{k})\exp(i\vec{k}\vec{x})$, which yields the eigenvalues of K_σ as

$$-\Phi_a^* K_\sigma \Phi_a = -\int \mathrm{d}^d\vec{x}\frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\frac{\mathrm{d}^d\vec{k}'}{(2\pi)^d}\left[\tilde{\Phi}_a^*(\vec{k}')(i^2\vec{k}\vec{k}' + (t+i\sigma))\tilde{\Phi}_a(\vec{k})e^{-i\vec{x}(\vec{k}-\vec{k}')}\right]$$
$$= -\int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\left[\tilde{\Phi}_a^*(\vec{k})(k^2 + (t+i\sigma))\tilde{\Phi}_a(\vec{k})\right],$$
$$= -\frac{1}{V}\sum_{\vec{k}} \tilde{\Phi}_{a,\vec{k}}^*(k^2 + (t+i\sigma))\tilde{\Phi}_{a,\vec{k}}$$

giving

$$\det K_\sigma \propto \prod_k (k^2 + (t+i\sigma)) = e^{\sum_k \log(k^2+(t+i\sigma))} = e^{V\int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\log(k^2+(t+i\sigma))}.$$

For the action $S_1[\Phi_1^*, \Phi_1, \sigma]$, we obtain in real space, assuming uniform field $|\Phi_1|^2$ and vanishing derivative, that

$$S_1[\Phi_1^*, \Phi_1, \sigma] = (t + i\sigma)V|\Phi_1|^2$$

By plugging this into the partition function equation, we obtain (up to prefactors)

$$\begin{aligned} Z &= \int \mathcal{D}\sigma\mathcal{D}\Phi_1^*\mathcal{D}\Phi_1 e^{-(t+i\sigma)V|\Phi_1|^2 - \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\left[V(N-1)\log(k^2+(t+i\sigma))\right] - \frac{NV}{2\lambda}\sigma^2} \\ &= \int \mathcal{D}\sigma\mathcal{D}\Phi_1^*\mathcal{D}\Phi_1 e^{-S[\Phi_1^*(k),\Phi_1(k),\sigma]} \end{aligned}$$

with action

$$S[\Phi_1^*(k), \Phi_1(k), \sigma] = (t + i\sigma)\,|\Phi_1|^2 + \frac{NV}{2\lambda}\sigma^2 + \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} V(N-1)\log(k^2 + (t + i\sigma))$$

After rotation $i\sigma \mapsto \sigma$ and rescaling $\Phi_1/\sqrt{N} \mapsto \Phi_1$, we have (with $N - 1 \approx N$)

$$S[\Phi_1^*(k), \Phi_1(k), \sigma] \approx NV\left((t + \sigma)\,|\Phi_1|^2 + \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\left[\log(k^2 + (t + \sigma))\right] - \frac{1}{2\lambda}\sigma^2\right)$$

For the saddle point approximation, we require

$$\begin{aligned} \frac{\delta S}{\delta \Phi_1^*} &= NV(t + \sigma)\Phi_1 = 0, \\ \frac{\delta S}{\delta \Phi_1} &= NV\Phi_1^*(t + \sigma) = 0, \\ \frac{\delta S}{\delta \sigma} &= NV\left((t + \sigma)\,|\Phi_1|^2 + \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\left[\frac{1}{k^2 + (t + \sigma)}\right] - \frac{1}{\lambda}\sigma\right) = 0. \end{aligned}$$

The first two conditions yield

$$(t + \sigma)\Phi_1 = 0.$$

By the third condition, we obtain

$$\sigma = \lambda \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\left(\frac{1}{k^2 + (t + \sigma)} + |\Phi_1|^2\right)$$

The (Landau) free energy density is then given by

$$\frac{f}{Nk_BT} = \frac{1}{NV}S[\Phi_1^*, \Phi_1, \sigma] = (t + \sigma)\,|\Phi_1|^2 - \frac{1}{2\lambda}\sigma^2 + \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\log(k^2 + t + \sigma).$$

(c) Show that the theory exhibits a phase transition for $d > 2$ at $t_\mathrm{c} = -\lambda \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{k^2}$ and that the inverse susceptibility $\chi^{-1} \propto t + \sigma$ satisfies in the disordered phase $\Phi_1 = 0$ the implicit equation

$$(t+\sigma)\left(1 + \lambda \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{k^2(k^2+t+\sigma)}\right) = t - t_\mathrm{c} \tag{20}$$

for $t > t_\mathrm{c}$. What happens for $d \leq 2$?

In the ordered phase, $\Phi_1 \neq 0$ and hence $t + \sigma = 0$. Therefore,

$$\sigma = \lambda \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \left(\frac{1}{k^2} + |\Phi_1|^2\right) = -t.$$

In the limit $|\Phi_1|^2 \to 0$ close to the phase transition, we obtain

$$t_\mathrm{c} = -\lambda \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{k^2}$$

In the disordered phase, we have

$$\sigma = \lambda \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{k^2 + (t+\sigma)}.$$

By partial fraction decomposition, we decompose

$$\frac{1}{k^2(k^2+t+\sigma)} = \frac{A}{k^2} + \frac{B}{k^2+t+\sigma},$$

finding $A = -B = 1/(t+\sigma)$. Hence, Eq. (20) is

$$\begin{aligned}(t+\sigma)&\left(1 + \frac{\lambda}{t+\sigma} \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{k^2} - \frac{\lambda}{t+\sigma} \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{k^2+t+\sigma}\right) \\ &= (t+\sigma)\left(1 - \frac{t_\mathrm{c}+\sigma}{t+\sigma}\right) = t - t_\mathrm{c}\end{aligned}$$

For $d < 2$, the critical tuning parameter t_c diverges, as

$$t_\mathrm{c} = -\lambda \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} \frac{1}{k^2} = -\lambda\Omega_d \int_0^\Lambda \mathrm{d}k \frac{k^{d-1}}{k^2} \propto \int_0^\Lambda \mathrm{d}k k^{d-3} \propto \begin{cases} \Lambda^{d-2} & d > 2 \\ \log\Lambda - \log 0 & d = 2 \\ \Lambda^{-2} - 0^{-2} & d = 1 \end{cases}.$$

(d) Assume an ultraviolet cutoff Λ in the integral over wavevectors and compute the scaling form of the susceptibility in the critical region $t + \sigma \to 0$ for (i) $d > 4$, (ii) $d = 4$, and (iii) $2 < d < 4$. Compare with the predictions from Landau theory for

the original model in Eq. (17).
Hint: (i) $\chi \propto |t-t_\mathrm{c}|^{-1}$, (ii) $\chi \propto \frac{\ln|t-t_\mathrm{c}|}{|t-t_\mathrm{c}|}$, (iii) $\chi \propto |t-t_\mathrm{c}|^{-2/(d-2)}$.

We take the following two Taylor series for granted:

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \qquad \log(1+x) = -\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}.$$

We begin by rewriting the implicit equation

$$\begin{aligned}
|t-t_\mathrm{c}| &= \chi^{-1}\left(1+\lambda\int_{|\vec{k}|<\Lambda}\frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\frac{1}{k^2}\frac{1}{k^2+\chi^{-1}}\right)\\
&= \chi^{-1}\left(1+C\chi\int_0^\Lambda \mathrm{d}k k^{d-3}\frac{1}{1+\chi k^2}\right)\\
&= \chi^{-1}\left(1+C\chi\int_0^\Lambda \mathrm{d}k k^{d-3}\sum_{n=0}^{\infty}(-\chi)^n k^{2n}\right)\\
&= \chi^{-1}+C\sum_{n=0}^{\infty}(-1)^n\chi^n\int_0^\Lambda \mathrm{d}k k^{d-3+2n},
\end{aligned}$$

for some constant C. If (i) $d = 4+x$, then

$$\begin{aligned}
|t-t_\mathrm{c}| &= \chi^{-1}+C\sum_{n=0}^{\infty}(-1)^n\chi^n\int_0^\Lambda \mathrm{d}k k^{1+x+2n}\\
&= \chi^{-1}+C\sum_{n=0}^{\infty}(-1)^n\chi^n\frac{\Lambda^{x+2n+2}}{x+2n+2}.
\end{aligned}$$

Up to leading order, this yields

$$\chi \propto |t-t_\mathrm{c}|^{-1}.$$

If (ii) $d = 4$, then

$$\begin{aligned}
|t-t_\mathrm{c}| &= \chi^{-1}+C\sum_{n=0}^{\infty}(-1)^n\chi^n\int_0^\Lambda \mathrm{d}k k^{1+2n}\\
&= \chi^{-1}+C\sum_{n=0}^{\infty}(-1)^n\chi^n\frac{\Lambda^{2(n+1)}}{2(n+1)}\\
&= \chi^{-1}-C'\chi^{-1}\sum_{n=0}^{\infty}(-1)^n\frac{(\chi\Lambda^2)^n}{n}\\
&= \chi^{-1}-C'\chi^{-1}\log(1+\chi\Lambda^2)\\
&\approx (1-C'\log\Lambda^2)\chi^{-1}-C'\chi^{-1}\log\chi,
\end{aligned}$$

as $\chi\Lambda^2 \gg 1$, which in leading order yields

$$\chi^{-1} \log \chi \propto |t - t_c|$$

Now, since $\sigma = -t_c + \mathcal{O}((t + \sigma))$, we have

$$\chi \propto \frac{\log |t + \sigma|}{|t - t_c|} \approx \frac{\log |t - t_c|}{|t - t_c|}.$$

Lastly for $2 < d = 3 < 4$, we have

$$\begin{aligned} |t - t_c| &= \chi^{-1} + C \sum_{n=0}^{\infty} (-1)^n \chi^n \int_0^{\Lambda} \mathrm{d}k k^{2n} \\ &= \chi^{-1} + C(\sqrt{\chi})^{-1} \sum_{n=0}^{\infty} (-1)^n (\sqrt{\chi})^{2n+1} \frac{\Lambda^{2n+1}}{2n+1} \\ &= \chi^{-1} + C\chi^{-1/2} \arctan(\chi\Lambda) \\ &= \chi^{-1} + C'\chi^{-1/2}, \end{aligned}$$

hence in leading order (with $\arctan(\chi\Lambda) \to \pi/2$)

$$\chi \propto |t - t_c|^{-2} = |t - t_c|^{-2/(d-2)}$$

16. Anisotropic perturbation to the $O(2)$ Wilson-Fisher fixed point

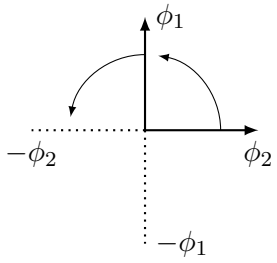
Consider the $O(2)$ model for the two-component boson field $\phi = (\phi_1, \phi_2)$ (with ϕ_1, ϕ_2 being real scalars) in the presence of an anisotropic perturbation:

$$S = \int \mathrm{d}^d x \left[\frac{1}{2}(\nabla\phi_1)^2 + \frac{1}{2}(\nabla\phi_2)^2 + \frac{r}{2}(\phi_1^2 + \phi_2^2) + \frac{u}{4!}(\phi_1^4 + \phi_2^4) + \frac{2v}{4!}\phi_1^2\phi_2^2 \right] \tag{21}$$

For $v \neq u$ the continuous $O(2)$ rotational symmetry is explicitly broken, but a residual $\mathbb{Z}_4$ symmetry (fourfold rotations by integer multiples of $\pi/2$) remains intact.

(a) Classify all possible symmetry-allowed operators with respect to their scaling dimension. Are there any relevant or marginal operators near $d = 4$ dimensions that have been omitted in Eq. (21)?

The unbroken $\mathbb{Z}_4$ symmetry corresponds to the following rotation:



yielding the 4 mappings

$$(\phi_1, \phi_2) \mapsto (-\phi_2, \phi_1) \mapsto (-\phi_1, -\phi_2) \mapsto (\phi_2, -\phi_1) \mapsto (\phi_1, \phi_2).$$

For any operator to be allowed by this symmetry, it has to

(1) act on an even number of fields, such that the minus-signs cancel, i.e. it has only terms such as $\phi_1^{2i}\phi_2^{2j}$ with $i, j \in \mathbb{N}$ and

(2) for each term with $\phi_1^i\phi_2^j$ a partner term $\phi_1^j\phi_2^i$ has to occur.

Now, let n be the number of spatial derivatives ∇ occurring in the operator, $2i$ ($2j$) the number of ϕ_1 (ϕ_2) terms and let g be the coupling of the operator. Then power-counting (in terms of inverse length units) yields

$$\begin{aligned} 0 = [S] &= [\mathrm{d}^d x] + [\nabla^n] + [\phi_1^{2i}] + [\phi_2^{2j}] + [g] \\ &= d[\mathrm{d}x] + n[\nabla] + 2i[\phi_1] + 2j[\phi_2] + [g]. \end{aligned}$$

Using $[\phi_i] = \frac{d-2}{2}$, $[\mathrm{d}x] = -1$ and $[\nabla] = 1$ yields the scaling dimension

$$[g] = d - n - (d-2)(i+j)$$

with the constraint $i + j \geq 1$ (at least one field in the term). We require $[g] \geq 0$ for the coupling to be marginal ($[g] = 0$) or relevant ($[g] > 0$). Consider first $i + j = 1$, i.e. one field term ϕ^2 occuring. This yields

$$[g] = 2 - n,$$

which is relevant for $n = 0, 1$ and marginal for $n = 2$. While the $n = 0$ and $n = 2$ case occur in Eq. (21), the $n = 1$ case has been omitted; this is however, because such a term $\phi(\nabla\phi)$ may be absorbed using a total derivative and hence does not change the action. For $i + j = 2$, we have

$$[g] = 4 - d - n = \epsilon - n,$$

which for $n = 0$ is relevant (marginal) for $\epsilon > 0$ ($\epsilon = 0$) and does indeed occur. For $i + j > 2$ close to $d = 4$ no other terms are relevant or marginal.

(b) Show that the one-loop RG flow of the suitably rescaled couplings in $d = 4 - \epsilon$ can

be written as

$$\frac{\mathrm{d}r}{\mathrm{d}\ln b} = 2r + \frac{1}{2}u + \frac{1}{6}v \tag{22}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\ln b} = \epsilon u - \frac{3}{2}u^2 - \frac{1}{6}v^2 \tag{23}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\ln b} = \epsilon v - \frac{2}{3}v^2 - uv \tag{24}$$

First, we perform the Fourier transformation $\phi_i(\vec{k}) = \int_0^\Lambda \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} e^{-i\vec{k}\vec{x}}\phi_i(\vec{x})$ with high-energy cutoff Λ. This yields the action

$$\begin{aligned} S = \int_0^\Lambda \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\Big[&\underbrace{\frac{1}{2}\phi_1(\vec{k})(k^2+r)\phi_1(-\vec{k})}_{\equiv S_{0,1}} + \underbrace{\frac{1}{2}\phi_2(\vec{k})(k^2+r)\phi_2(-\vec{k})}_{\equiv S_{0,2}}\Big] \\ &+ \underbrace{\frac{u}{4!}\int \frac{\mathrm{d}^d\vec{k}_1\mathrm{d}^d\vec{k}_2\mathrm{d}^d\vec{k}_3}{(2\pi)^{3d}}\phi_1(\vec{k}_1)\phi_1(\vec{k}_2)\phi_1(\vec{k}_3)\phi_1(-\vec{k}_1-\vec{k}_2-\vec{k}_3)}_{S_{\mathrm{int},1}} + \underbrace{(1\leftrightarrow 2)}_{S_{\mathrm{int},2}} \\ &+ \underbrace{\frac{2v}{4!}\int \frac{\mathrm{d}^d\vec{k}_1\mathrm{d}^d\vec{k}_2\mathrm{d}^d\vec{k}_3}{(2\pi)^{3d}}\phi_1(\vec{k}_1)\phi_1(\vec{k}_2)\phi_2(\vec{k}_3)\phi_2(-\vec{k}_1-\vec{k}_2-\vec{k}_3)}_{S_{\mathrm{mix}}} \end{aligned}$$

We begin the RG by eliminating the high-energy modes through integration. Let $\phi_1^<$ ($\phi_2^<$) and $\phi_1^>$ ($\phi_2^>$) be the high and low energy modes of ϕ_1 (ϕ_2) respectively. While for $S_{0,i} + S_{\mathrm{int},i}$ we use the results from the lecture, for S_{mix}, we need to derive the contribution. Hence,

$$\begin{aligned} S_{\mathrm{mix}} &= \frac{2v}{4!}\int \phi_1\phi_1\phi_2\phi_2 \\ &= \frac{2v}{4!}\Bigg(\int_0^{\Lambda/b} \phi_1^<\phi_1^<\phi_2^<\phi_2^< + \int_{\Lambda/b}^{\Lambda} \phi_1^>\phi_1^>\phi_2^>\phi_2^> \\ &\quad + \int_0^\Lambda (\phi_1^<\phi_1^<\phi_2^>\phi_2^> + \phi_1^<\phi_1^>\phi_2^<\phi_2^> + \phi_1^>\phi_1^>\phi_2^<\phi_2^<)\Bigg) \end{aligned}$$

Now, taking the average with respect to the Gaussian action of the high-energy modes

$$\langle\ldots\rangle_{0>} = \frac{1}{Z_0}\int \mathcal{D}\phi_1^>\mathcal{D}\phi_2^>(\ldots)e^{-S_{0,1}^>[\phi_1^>]-S_{0,2}^>[\phi_2^>]},$$

we find

$$\begin{aligned} S_{\mathrm{mix}} = \frac{2v}{4!}\Bigg(&\int_0^{\Lambda/b} \underbrace{\langle\phi_1^<\phi_1^<\phi_2^<\phi_2^<\rangle_{0>}}_{\text{tree level}} + \int_{\Lambda/b}^{\Lambda} \underbrace{\langle\phi_1^>\phi_1^>\phi_2^>\phi_2^>\rangle_{0>}}_{\text{vacuum}\to\text{const.}} \\ &+ \int_0^\Lambda (\langle\phi_1^<\phi_1^<\phi_2^>\phi_2^>\rangle_{0>} + \langle\phi_1^<\phi_1^>\phi_2^<\phi_2^>\rangle_{0>} + \langle\phi_1^>\phi_1^>\phi_2^<\phi_2^<\rangle_{0>})\Bigg) \end{aligned}$$

In the Gaussian theory, ϕ_1 and ϕ_2 do not mix, hence we obtain using Wick's theorem

$$\langle \phi_1^< \phi_1^< \phi_2^> \phi_2^> \rangle_{0>} = \langle \phi_1^< \phi_1^< \rangle_{0>} \langle \phi_2^> \phi_2^> \rangle_{0>} = \langle \phi_1^< \phi_1^< \rangle_{0>} \int_{\Lambda/b}^{\Lambda} \frac{1}{k^2+r},$$
$$\langle \phi_1^< \phi_1^> \phi_2^< \phi_2^> \rangle_{0>} = 0,$$
$$\langle \phi_1^> \phi_1^> \phi_2^< \phi_2^< \rangle_{0>} = \langle \phi_1^> \phi_1^> \rangle_{0>} \langle \phi_2^< \phi_2^< \rangle_{0>} = \langle \phi_2^< \phi_2^< \rangle_{0>} \int_{\Lambda/b}^{\Lambda} \frac{1}{k^2+r}.$$

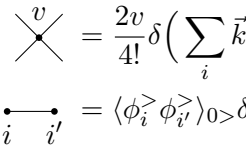
In the lecture, we solved the integral

$$\int_{\Lambda/b}^{\Lambda} \frac{1}{k^2+r} = \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{\Lambda^2+r} \ln b + \mathcal{O}(\ln^2 b).$$

Hence, in leading order of v, this does not provide the RG flow equations and we need to consider the leading order correction, given by the Feynman diagram

$$= (-1)\frac{1}{2!}\left(\frac{-2v}{4!}\right)^2 (\phi_1\phi_1\phi_2\phi_2)(\phi_1\phi_1\phi_2\phi_2)$$

with the Feynman rules



$$= \frac{2v}{4!}\delta\Big(\sum_i \vec{k}_i\Big)\delta_{n_1,2}\delta_{n_2,2}, \qquad = \frac{u}{4!}\delta\Big(\sum_i \vec{k}_i\Big)(\delta_{n_1,4}+\delta_{n_2,4}),$$
$$= \langle \phi_i^> \phi_{i'}^> \rangle_{0>}\delta_{ii'}, \qquad = \phi_i^<.$$

Hence

$$(\phi_1\phi_1\phi_2\phi_2)(\phi_1\phi_1\phi_2\phi_2) = \Big(2(\phi_1^<)^4 + (2\times 2)^2(\phi_1^<)^2(\phi_2^<)^2 + 2(\phi_2^<)^4\Big) I_r,$$

where $I_r = \int_{\Lambda/b}^{\Lambda} \frac{1}{(k^2+r)^2}$, the factor $2(\phi_i^<)^4$ is obtained by the two permutations of $(\phi_j)^2$ and 2×2 is obtained from the ways to choose the surviving $\phi_i^<$ from the two pairs $(\phi_i)^2$. We may compute I_r to first order in $\ln b$ by using the Leibniz rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(t,x)\mathrm{d}x = f(t,a(t))a'(t) + f(t,b(t))b'(t) + \int_{a(t)}^{b(t)} \partial_t f(t,x)\mathrm{d}x,$$

which yields for the Taylor expansion

$$I_r = 0 + \left.\frac{\mathrm{d}I_r}{\mathrm{d}\ln b}\right|_{\ln b=0} \ln b + \mathcal{O}(\ln^2 b)$$

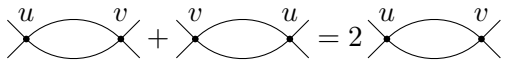
with

$$\frac{\mathrm{d}I_r}{\mathrm{d}\ln b} = \frac{\mathrm{d}}{\mathrm{d}\ln b}\frac{S_d}{(2\pi)^d}\int_{\Lambda e^{-\ln b}}^{\Lambda} \mathrm{d}k \frac{k^{d-1}}{(k^2+r)^2} = \frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{(\Lambda^2+r)^2},$$

finally

$$I_r = \frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{(\Lambda^2+r)^2}\ln b + \mathcal{O}(\ln^2 b)$$

Lastly, we need to consider the Feynman diagram coming from uv interactions, namely



$$\begin{aligned}
&= 2(-1)\frac{1}{2!}\frac{(-2v)}{4!}\frac{(-u)}{4!}(\phi_i^4)(\phi_i^2\phi_j^2)\\
&= 2(-1)\frac{1}{2!}\frac{(-2v)}{4!}\frac{(-u)}{4!}\binom{4}{2}(2\times 2)(\phi_1^<)^2(\phi_2^<)^2 I_r,
\end{aligned}$$

where $\binom{4}{2}$ comes from choosing the elements of $(\phi_i)^4$ and one 2 comes from permuting ϕ_1 and ϕ_2, while the other is obtained from the two permutations of the contracted fields. Now, the action after integrating the high-energy modes is

$$\begin{aligned}
S^< = S^<_{0,i} &+ \frac{u}{4!}\left[\int_0^{\Lambda/b}(\phi_i^<)^4 + \binom{4}{2}\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}\ln b\int_0^{\Lambda/b}(\phi_i^<)^2 - \frac{3}{2}u\int_0^{\Lambda/b}(\phi_i^<)^4 I_r\right]\\
&+ \frac{2v}{4!}\left[\int_0^{\Lambda/b}(\phi_1^<)^2(\phi_2^<)^2 + \frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}\ln b\int_0^{\Lambda/b}(\phi_i^<)^2\right.\\
&\left. - \frac{1}{2!}\frac{2v}{4!}\int_0^{\Lambda/b}\left(2(\phi_i^<)^4 + 4^2(\phi_1^<)^2(\phi_2^<)^2\right) I_r\right] - 4\frac{2v}{4!}\frac{u}{4!}\binom{4}{2}\int_0^{\Lambda/b}(\phi_1^<)^2(\phi_2^<)^2 I_r.
\end{aligned}$$

We can reorder this, to obtain the same form as the original action finding (with $C \equiv \frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}\ln b$ and $I_r = \frac{C}{\Lambda^2+r}$)

$$\begin{aligned}
S^< = \int &\frac{1}{2}\phi_i^<\left(k^2 + r + \frac{C}{2}u + \frac{C}{6}v\right)\phi_i^< + \frac{1}{4!}\left(u - \frac{v^2}{6}\frac{C}{\Lambda^2+r} - \frac{3}{2}u^2\frac{C}{\Lambda^2+r}\right)(\phi_i^<)^4\\
&+ \frac{2}{4!}\left(v - \frac{2}{3}v^2\frac{C}{\Lambda^2+r} - uv\frac{C}{\Lambda^2+r}\right)(\phi_1^<)^2(\phi_2^<)^2
\end{aligned}$$

Next, we rescale the momenta $\vec{k}' = b\vec{k}$, such yielding

$$\begin{aligned}
S^< = &\int \frac{1}{2}\phi_i^< b^{-d}\left(b^{-2}k'^2 + r + \frac{C}{2}u + \frac{C}{6}v\right)\phi_i^<\\
&+ \int b^{-3d}\frac{1}{4!}\left(u - \frac{v^2}{6}\frac{C}{\Lambda^2+r} - \frac{3}{2}u^2\frac{C}{\Lambda^2+r}\right)(\phi_i^<)^4\\
&+ \int b^{-3d}\frac{2}{4!}\left(v - \frac{2}{3}v^2\frac{C}{\Lambda^2+r} - uv\frac{C}{\Lambda^2+r}\right)(\phi_1^<)^2(\phi_2^<)^2.
\end{aligned}$$

After rescaling the fields $\phi'(\vec{k}') = b^{-\frac{d+2}{2}}\phi^<(\vec{k}'/b)$ to obtain an invariant form of the k'^2 term, we find the RG action

$$\begin{aligned}
S^< = &\int \frac{1}{2}\phi_i^< \left(k'^2 + b^2 r + b^2\frac{C}{2}u + b^2\frac{C}{6}v\right)\phi_i^< \\
&+ \int b^{4-d}\frac{1}{4!}\left(u - \frac{v^2}{6}\frac{C}{\Lambda^2+r} - \frac{3}{2}u^2\frac{C}{\Lambda^2+r}\right)(\phi_i^<)^4 \\
&+ \int b^{4-d}\frac{2}{4!}\left(v - \frac{2}{3}v^2\frac{C}{\Lambda^2+r} - uv\frac{C}{\Lambda^2+r}\right)(\phi_1^<)^2(\phi_2^<)^2.
\end{aligned}$$

This yields the three renormalized couplings (having removed the $\ln b$ contribution from C, i.e. $C \mapsto C\ln b$)

$$\begin{aligned}
r' &= b^2 r + b^2 C\ln b\frac{1}{2}u + b^2 C\ln b\frac{1}{6}v, \\
u' &= b^{4-d}u - b^{4-d}\ln b\frac{C}{\Lambda^2+r}\frac{1}{6}v^2 - b^{4-d}\ln b\frac{C}{\Lambda^2+r}\frac{3}{2}u^2, \\
v' &= b^{4-d}v - b^{4-d}\ln b\frac{C}{\Lambda^2+r}\frac{2}{3}v^2 - b^{4-d}\ln b\frac{C}{\Lambda^2+r}uv.
\end{aligned}$$

We now use $b^x\ln(b) = e^{x\ln b}\ln b = (1 + x\ln b + \ldots)\ln b = \ln b + \mathcal{O}(\ln^2 b)$, plug in $\epsilon = 4-d$, redefine $\tilde{r} = \Lambda^{-2}r$, $\tilde{v} = \frac{S_d}{(2\pi)^d}\Lambda^{d-4}v$ and $\tilde{u} = \frac{S_d}{(2\pi)^d}\Lambda^{d-4}u$, finding

$$\begin{aligned}
\tilde{r}' &= e^{2\ln b}\tilde{r} + \frac{1}{2}\tilde{u}\frac{1}{1+\tilde{r}^2}\ln b + \frac{1}{6}\tilde{v}\frac{1}{1+\tilde{r}^2}\ln b, \\
\tilde{u}' &= e^{\epsilon\ln b}\tilde{u} - \frac{1}{6}\tilde{v}^2\frac{1}{(1+\tilde{r}^2)^2}\ln b - \frac{3}{2}\tilde{u}^2\frac{1}{(1+\tilde{r}^2)^2}\ln b, \\
\tilde{v}' &= e^{\epsilon\ln b}\tilde{v} - \frac{2}{3}\tilde{v}^2\frac{1}{(1+\tilde{r}^2)^2}\ln b - \tilde{v}\tilde{u}\frac{1}{(1+\tilde{r}^2)^2}\ln b.
\end{aligned}$$

Which for $\frac{1}{1+\tilde{r}^2} \approx 1$ and upon differentiation with respect to $\ln b$, yields the RG-flow equations in Eqs. (22) to (24).

(c) Show that these equations reduce to the expected flow equation of the $O(2)$ model in the limit $u = v$.

Let $v = u + \delta$, then the RG flow-equations in Eqs. (22) to (24) read

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\ln b} &= 2r + \frac{2}{3}u + \frac{1}{6}\delta, \\
\frac{\mathrm{d}u}{\mathrm{d}\ln b} &= \epsilon u - \frac{5}{3}u^2 + \frac{1}{3}u\delta + \mathcal{O}(\delta^2), \\
\frac{\mathrm{d}u}{\mathrm{d}\ln b} + \frac{\mathrm{d}\delta}{\mathrm{d}\ln b} &= \epsilon u - \frac{5}{3}u^2 - \left(\frac{5}{3}u + \epsilon\right)\delta
\end{aligned}$$

For $\delta \to 0$, we retrieve

$$\frac{\mathrm{d}r}{\mathrm{d}\ln b} = 2r + \frac{2+2}{6}u, \qquad \frac{\mathrm{d}u}{\mathrm{d}\ln b} = \epsilon u - \frac{2+8}{6}u^2,$$

which are the expected flow equation for the $O(N=2)$ model as derived in the lecture.

(d) Determine the linearized RG flow in the vicinity of the Wilson-Fisher fixed point at $r = r^*$ and $u = v = u^*$:

$$\frac{\mathrm{d}\delta g_i}{\mathrm{d}\ln b} = \sum_{j=1}^{3} B_{ij}\delta g_j + \mathcal{O}(\delta g^2), \qquad \delta g_i = g_i - g_i^*, \tag{25}$$

with the "stability matrix" $B_{ij} = \frac{\partial(\mathrm{d}g_i/\mathrm{d}\ln b)}{\partial g_j}\big|_{g=g^*}$ and $(g_i) = (r, u, v)$. Is the $\mathbb{Z}_4$ anisotropy $\propto u - v$ relevant or irrelevant at the Wilson-Fisher fixed point? What is the corresponding eigenvalue of the stability matrix?

From the RG flow equations, we have

$$u^* = -3r^* = \frac{3}{5}\epsilon$$

and find

$$\begin{aligned}
\frac{\mathrm{d}\delta r}{\mathrm{d}\ln b} &= 2(r^* + \delta r) + \frac{1}{2}(u^* + \delta u) + \frac{1}{6}(v + \delta v) = 2\delta r + \frac{1}{2}\delta u + \frac{1}{6}\delta v \\
\frac{\mathrm{d}\delta u}{\mathrm{d}\ln b} &= \epsilon(u^* + \delta u) - \frac{3}{2}(u^* + \delta u)^2 - \frac{1}{6}(u^* + \delta v)^2 = \left(1 - \frac{9}{5}\right)\epsilon\delta u - \frac{1}{5}\epsilon\delta v + \mathcal{O}(\delta^2), \\
\frac{\mathrm{d}\delta v}{\mathrm{d}\ln b} &= \epsilon(u^* + \delta v) - \frac{2}{3}(u^* + \delta v)^2 - (u^* + \delta u)(u^* + \delta v) \\
&= \left(1 - \frac{4}{5} - \frac{3}{5}\right)\epsilon\delta v - \frac{3}{5}\epsilon\delta u + \mathcal{O}(\delta^2),
\end{aligned}$$

hence the stability matrix is

$$B = \begin{pmatrix} 2 & \frac{1}{2} & \frac{1}{6} \\ 0 & -\frac{4}{5}\epsilon & -\frac{1}{5}\epsilon \\ 0 & -\frac{3}{5}\epsilon & -\frac{2}{5}\epsilon \end{pmatrix}$$

For the $\mathbb{Z}_4$ anisotropy, define $a = u - v$, hence $\delta a = \delta u - \delta v$ and

$$\frac{\mathrm{d}\delta a}{\mathrm{d}\ln b} = \frac{\mathrm{d}\delta u}{\mathrm{d}\ln b} - \frac{\mathrm{d}\delta v}{\mathrm{d}\ln b} = -\frac{1}{5}\epsilon\delta u + \frac{1}{5}\epsilon\delta v = -\frac{1}{5}\epsilon\delta a,$$

i.e. the anisotropy has scaling dimension $-\epsilon/5$, which for $\epsilon > 0$ is negative and therefore is irrelevant. The eigenvalues of the stability matrix B are

$$\lambda_1 = 2, \qquad \lambda_2 = -\epsilon, \qquad \lambda_3 = -\epsilon/5$$

with corresponding eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{v}_1 = \begin{pmatrix} -\frac{2}{3(2+\epsilon)} \\ 1 \\ 1 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix},$$

The eigenvalue and eigenvector corresponding to the $\mathbb{Z}_4$ anisotropy are λ_3 and $\vec{v}_3$.

17. Interchange of limits in the classical Ising chain

Consider a classical Ising chain with M sites and nearest-neighbor exchange ($K > 0$):

$$H = -\sum_i K\sigma_i\sigma_{i+1}. \tag{26}$$

(a) Write down the partition function

$$Z \equiv \sum_{\{\sigma_i=\pm 1\}} e^{-H} \tag{27}$$

in transfer-matrix representation.

We assume periodic boundary conditions, i.e. $\sigma_M = \sigma_0$ and hence find for the partition function

$$Z = \sum_{\{\sigma_i\}} e^{-H} = \sum_{\{\sigma_i\}} e^{K\sum_i \sigma_i\sigma_{i+1}} = \sum_{\{\sigma_i\}} \prod_i e^{K\sigma_i\sigma_{i+1}} = \sum_{\{\sigma_i\}} \prod_i T_{\sigma_i,\sigma_{i+1}},$$

where we defined the transfer-matrix

$$(T)_{\sigma_i,\sigma_{i+1}} = e^{K\sigma_i\sigma_{i+1}} \implies T = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix}.$$

Then sums correspond to matrix products, such that

$$\begin{aligned} Z &= \sum_{\sigma_0}\sum_{\sigma_1} T_{\sigma_0,\sigma_1} \sum_{\sigma_2} T_{\sigma_1,\sigma_2} \cdots \sum_{\sigma_{M-1}} T_{\sigma_{M-2},\sigma_{M-1}} T_{\sigma_{M-1},\sigma_0} \\ &= \sum_{\sigma_0} (T^{M-1})_{\sigma_0,\sigma_0} = \operatorname{tr} T^{M-1}. \end{aligned}$$

(b) Evaluate Z as well as the spin-spin correlation function $\langle\sigma_i\sigma_0\rangle$ exactly.

To evaluate Z, we diagonalize T, finding the eigenvalues

$$\lambda_1 = e^K + e^{-K}, \quad \lambda_2 = e^K - e^{-K}$$

and note that

$$Z = \operatorname{tr} T^{M-1} = \lambda_1^{M-1} + \lambda_2^{M-1} = \lambda_1^{M-1}\left(1 + x^{M-1}\right)$$

with $x = \lambda_2/\lambda_1$.
Now, for the correlator we find

$$\begin{aligned}\langle \sigma_i \sigma_0 \rangle &= \frac{1}{Z} \sum_{\{\sigma\}} \sigma_i \sigma_0 e^{-H} \\ &= \frac{1}{Z} \sum_{\sigma_0, \sigma_i} \sigma_{\sigma_0,\sigma_0} (T^i)_{\sigma_0,\sigma_i} \sigma_{\sigma_i,\sigma_i} (T^{M-i-1})_{\sigma_i,\sigma_0} \\ &= \frac{1}{Z} \operatorname{tr} \sigma T^i \sigma T^{M-i-1}\end{aligned}$$

with $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We diagonalize T as

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

such that

$$T^n = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & \\ & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \lambda_1^n \begin{pmatrix} 1 + x^n & 1 - x^n \\ 1 - x^n & 1 + x^n \end{pmatrix}$$

Then,

$$\langle \sigma_i \sigma_0 \rangle = \frac{1}{Z} \operatorname{tr} \left[\frac{\lambda_1^{M-1}}{4} 2 \begin{pmatrix} x^i + x^{M-i-1} & x^i - x^{M-i-1} \\ x^i - x^{M-i-1} & x^i + x^{M-i-1} \end{pmatrix} \right] = \frac{x^i + x^{M-i-1}}{1 + x^{M-1}}.$$

(c) Investigate now two possible routes to obtain the correlation function in the limit of large K (small T).

(i) Approximate $\langle \sigma_i \sigma_0 \rangle$ first for large K and then take the limit $M \to \infty$.

For large K, we need to approximate the eigenvalue ratio x first, finding

$$x = \frac{\lambda_2}{\lambda_1} = \frac{e^K - e^{-K}}{e^K + e^{-K}} = \frac{e^{2K} - 1}{e^{2K} + 1} = 1 - \frac{2}{e^{2K} + 1} \approx 1 - 2e^{-2K}.$$

Therefore, for the correlation function $\langle \sigma_i \sigma_0 \rangle$ we obtain

$$\begin{aligned}\langle \sigma_i \sigma_0 \rangle &= \frac{(1 - 2e^{-2K})^i + (1 - 2e^{-2K})^{M-i-1}}{1 + (1 - 2e^{-2K})^{M-1}} \\ &\approx \frac{2 - 2ie^{-2K} - 2(M - i - 1)e^{-2K}}{2 - 2(M-1)e^{-2K}} = 1,\end{aligned}$$

independent of M.

(i) Take first the limit $M \to \infty$ and then approximate $\langle \sigma_i \sigma_0 \rangle$ for large K.

Expanding the chain first, i.e. taking $M \to \infty$ yields

$$\langle \sigma_i \sigma_0 \rangle \approx x^i = e^{i \ln x} = e^{i \ln(1-2e^{-2K})} \approx e^{-2ie^{-2K}} \equiv e^{-\frac{i}{\xi}}$$

with correlation length $\xi = e^{2K}/2$, which is $\xi = (a/2)e^{\Delta}$ with $a = 1$ the lattice spacing.

(b) Why are the results different? Explain which different physical situations the two routes correspond to.
Hint: Determine the energy Δ required to create a domain wall between a region with all spins up and a region with all spins down, and think in terms of domain walls.

We may rewrite the Hamiltonian as

$$H = -KM - \sum_i K(\sigma_i \sigma_{i+1} - 1) \equiv E_0 + \Delta N_d,$$

with $E_0 = -KM$ the energy of the ground state, $\Delta = 2K$ the energy needed to create the domain wall and N_d the number of domain walls. Now, by first sending $K \to \infty$, we increase Δ and render the formation of domain walls extremely costly. Now, keeping the energy-scale fixed and increasing the system size M, we have $\Delta/E_0 \propto 1/M$, i.e. domain wall excitations away from the ground state are associated with small energies compared to the whole system and do not get suppressed immediately.

18. Classical Ising chain with next-nearest neighbor interactions

Consider the infinite ($M \to \infty$) classical Ising chain with first and second neighbor exchange ($K_1, K_2 > 0$)

$$H = -\sum_i (K_1 \sigma_i \sigma_{i+1} + K_2 \sigma_i \sigma_{i+2}). \tag{28}$$

(a) Write down the partition function Z as a transfer-matrix product.
Hint: Think of the model in terms of "superspins" with 4 states, each of which represents a block of two neighboring Ising spins.

Let $|s_i\rangle = |\sigma_i, \sigma_{i+1}\rangle$ be such a superspin with $|++\rangle$, $|+-\rangle$, $|-+\rangle$ and $|--\rangle$ the basis order. The Hamiltonian can be rewritten as a sum of local terms in this super spin formalism as $H = -\sum_i h_{i,i+1}$ with

$$h_{i,i+1} = \frac{K_1}{2}(\sigma_i \sigma_{i+1} + \sigma_{i+1} \sigma_{i+2}) + K_2 \sigma_i \sigma_{i+2}$$

Then the transfer-matrix is

$$(T)_{i,i+1} = \langle s_i | e^{-h_{i,i+1}} | s_{i+1} \rangle = \begin{pmatrix} e^{K_1+K_2} & e^{-K_2} & 0 & 0 \\ 0 & 0 & e^{-K_1+K_2} & e^{-K_2} \\ e^{-K_2} & e^{-K_1+K_2} & 0 & 0 \\ 0 & 0 & e^{-K_2} & e^{K_1+K_2} \end{pmatrix}$$

Then the partition function Z is

$$Z = \operatorname{tr} T^M.$$

(b) Is it possible to diagonalize the transfer matrices using a unitary transformation?

Yes, the matrix may be diagonalized with the following eigenvalues

$$\lambda_{1\pm} = e^{K_2}(\cosh K_1 \pm \sqrt{\sinh^2 K_1 + e^{-4K_2}}),$$
$$\lambda_{2\pm} = e^{K_2}(\sinh K_1 \pm \sqrt{\cosh^2 K_1 - e^{-4K_2}}).$$

The larger two eigenvalues are λ_{1+} and λ_{2+} respectively.

(c) Determine the energy Δ required to create a domain wall.

We apply the same rewrite as before, finding

$$\begin{aligned} H &= -K_1 M - K_2 M - \sum_i (K_1(\sigma_i \sigma_{i+1} - 1) + K_2(\sigma_i \sigma_{i+2} - 1)) \\ &= -K_1 M - K_2 M + \underbrace{2(K_1 + 2K_2)}_{\equiv \Delta} N_d, \end{aligned}$$

because two next-nearest neighbor terms give a contribution per domain wall (assuming that the domain comprises more than one spins). Hence, the energy to create one domain wall is

$$\Delta = 2(K_1 + 2K_2)$$

(d) Determine the correlation length ξ of the model for large K_1, K_2, and show that $\xi = (a/2)e^{\Delta}$ (a is the lattice spacing).
Hint: Recall the lesson learned from Problem 1.

In Problem 1, we obtained first taking $M \to \infty$, that the correlation function scales as

$$\langle \sigma_i \sigma_0 \rangle \propto x^i$$

with $x = \lambda_2/\lambda_1$ the ratio between the largest (λ_1) and second largest (λ_2) eigen-

values. We find

$$x = \frac{\lambda_2}{\lambda_1} = \frac{\sinh K_1 + \cosh K_1 \sqrt{1 - \frac{e^{-4K_2}}{\cosh^2 K_1}}}{\cosh K_1 + \sinh K_1 \sqrt{1 + \frac{e^{-4K_2}}{\sinh^2 K_1}}}$$

and approximate $\sqrt{1+x} \approx 1 + \frac{x}{2}$ to obtain for large K_1 and K_2

$$x = \frac{\sinh K_1 + \cosh K_1 - \frac{e^{-4K_2}}{2\cosh K_1}}{\cosh K_1 + \sinh K_1 + \frac{e^{-4K_2}}{2\sinh K_1}} \approx \frac{1 - e^{-4K_2 - 2K_1}}{1 + e^{-4K_2 - 2K_1}} \approx 1 - 2e^{-\Delta}.$$

With the same form of the ratio x, we obtain the same correlation length

$$\xi = (a/2)e^{\Delta}$$

as before in Problem 1.

19. Relation between energy gap and correlation length

We wish to show now that the relationship $\xi = (a/2)e^{\Delta}$ holds quite generally, i.e. independently of the model and, to some extent, temperature. To this end, we think of the spin configurations in terms of domain walls and assume the domain walls to be statistically uncorrelated from each other (i.e., we neglect possible interactions between the domain walls).

(a) Argue that the density of domain walls is given by $\rho = (1/a)e^{-\Delta}$. Consequently, it is sufficient to show $\xi = 1/(2\rho)$.

The density of domain walls is the ratio between the number of domain walls N and the total length of the chain $L = Ma$, i.e.

$$\rho = \frac{N}{Ma} = \frac{1}{a}\frac{N}{M}.$$

The Hamiltonian $H = -E_0 + \Delta N$ of the spin models depends only on the ground state energy $E_0 = \epsilon M$ and the number of domain walls. Therefore, by noting that there are approximately M^N ways to assign the N domain walls to the spin chain (having assumed that their positioning is independent and that $M \gg N$, such that no overlap occurs), the partition function is

$$Z = \sum_N M^N e^{-H(N)} = \sum_N M^N e^{E_0 - \Delta N} = e^{E_0} \sum_N (Me^{-\Delta})^N = \frac{e^{E_0}}{1 - Me^{-\Delta}}.$$

The expected number of domain walls is similarly

$$\langle N \rangle = \frac{1}{Z} \sum_N N M^N e^{-H(N)} = \frac{1}{Z}(-\partial_\Delta) Z = -\partial_\Delta \ln Z.$$

With $\ln Z - E_0 = -\ln(1 - Me^{-\Delta}) \approx Me^{-\Delta}$ this yields

$$\langle N \rangle \approx Me^{-\Delta},$$

such that

$$\rho = \frac{e^{-\Delta}}{a}.$$

(b) Consider a long chain of length $Ma \gg \xi$ with $N = \rho Ma$ domain walls. The probability that any given domain wall is between 0 and $x > 0$ is $q = x/(Ma)$. Use the statistical independence of the domain walls to argue that

$$\langle \sigma(x)\sigma(0) \rangle = \sum_{j=0}^{N} (-1)^j q^j (1-q)^{N-j} \frac{N!}{j!(N-j)!}. \tag{29}$$

The probability p_j that there are j domain walls between 0 and x is binomially distributed, as the domain walls are independent and therefore

$$p_j = \binom{N}{j} q^j (1-q)^{N-j}.$$

Furthermore, every time a domain wall occurs, the sign of the correlator flips from $+1$ to -1 and back; i.e. for zero domain wall occurrences, the correlator if $+1$, for one occurrence -1 and so on. Therefore, the expected value of the correlator is

$$\langle \sigma(x)\sigma(0) \rangle = \sum_{j=0}^{N} (-1)^j p_j,$$

which is exactly Eq. (29).

(c) Evaluate the above expression in the limit $N, M \to \infty$, while $\rho = N/(Ma)$ is finite, to show the desired result.

By the Binomial theorem

$$(x+y)^N = \sum_{j=0}^{N} \binom{N}{j} x^j y^{N-j},$$

we find that Eq. (29) may be written as

$$\langle \sigma(x)\sigma(0) \rangle = (-q + 1 - q)^N = \left(1 - \frac{2x}{Ma}\right)^N = \left(1 - \frac{2x\rho}{N}\right)^N \xrightarrow{N\to\infty} e^{-2x\rho}.$$

Hence, the correlation length is

$$\xi = \frac{1}{2\rho} = (a/2)e^{\Delta},$$

as desired.

20. Quantum Ising chain with second-neighbor exchange

Consider a quantum Ising chain with second-neighbor exchange in a transverse field,

$$H_{\mathrm{I}} = -\sum_n (J\sigma_n^z\sigma_{n+1}^z + J_2\sigma_n^z\sigma_{n+2}^z + Jg\sigma_n^x). \tag{30}$$

Here, the spin-1/2 operators are represented by Pauli matrices σ_n^x and σ_z^n that fulfill the algebra:

$$\begin{aligned} \sigma_n^z\sigma_n^z &= \sigma_n^x\sigma_n^x = 1, \\ \sigma_n^z\sigma_n^x &= -\sigma_n^x\sigma_n^z, \\ \sigma_n^z\sigma_m^x &= \sigma_m^x\sigma_m^z, \qquad \text{for } m \neq n. \end{aligned} \tag{31}$$

(a) Determine the dispersion relation of a domain-wall excitation to lowest order in g.

For $g = 0$ the groundstate manifold of the Hamiltonian H_{I} is spanned by the fully polarized states

$$|\Uparrow\rangle = |\cdots\uparrow\uparrow\uparrow\cdots\rangle, \qquad |\Downarrow\rangle = |\cdots\downarrow\downarrow\downarrow\cdots\rangle.$$

Let $|E_0\rangle$ be the groundstate with energy E_0 chosen by the system. Then, the domain-wall excitations are created by flipping all spins up to some point, i.e., their creation operator is

$$w_j^\dagger = \prod_{n=j}^{\infty} \sigma_n^x$$

with the excited states $|j\rangle = w_j^\dagger|E_0\rangle$.
Now, upon turning on g, these excited states mix because of the action of σ_n^x close to the domain-wall. In first order of perturbation theory (for small g, therefore neglecting spin flip excitations away from the boundary), the energy correction is given by

$$H_{\mathrm{I}}|j\rangle = E_0|j\rangle + 2J|j\rangle + 4J_2|j\rangle - Jg|j-1\rangle - Jg|j+1\rangle + \mathcal{O}(g^2).$$

This Hamiltonian may be diagonalized by inserting the Fourier transformation $|j\rangle = \frac{1}{\sqrt{N}}\sum_k e^{ijk}|k\rangle$, such that

$$\frac{1}{\sqrt{N}}\sum_k e^{ijk}(H_{\mathrm{I}} - E_0)|k\rangle = \frac{1}{\sqrt{N}}\sum_k e^{ijk}(2J + 4J_2 - Jg(e^{ik} + e^{-ik})) + \mathcal{O}(g^2).$$

By comparing coefficients, we find the dispersion relation of a domain-wall excitation to lowest order in g as

$$\epsilon(k) = 2J + 4J_2 - 2Jg\cos k \approx \Delta_1 - Jgk^2$$

with $\Delta_1 = 2J(1-g) + 4J_2$.

(b) Determine the dispersion relation of a flipped-spin excitation in the limit $g \gg 1$.

Similarly, for $g \gg 1$ the groundstate is fully polarized in the positive x direction, i.e., it is

$$|\Rightarrow\rangle = |\cdots \rightarrow\rightarrow\rightarrow \cdots\rangle,$$

with energy $E_{\Rightarrow}$. The flipped-spin excitations are created by $f_j^\dagger = \sigma_j^z$, such that the excited states are $|j\rangle = f_j^\dagger|\Rightarrow\rangle$. We again find the energy correction

$$H_1|j\rangle = E_0|j\rangle + 2Jg|j\rangle - J|j-1\rangle - J|j+1\rangle - J_2|j-2\rangle - J_2|j+2\rangle + \mathcal{O}(g^2),$$

such that the dispersion relation obtained from Fourier transform is

$$\epsilon(k) = 2Jg - 2J\cos k - 2J_2\cos 2k \approx \Delta_2 - (J + 4J_2)k^2,$$

with $\Delta_2 = 2J(g-1)$.

(c) Interpret the results.

First, consider the case with $J_2 = 0$.

- There, both excitations are gapped, with energy gaps Δ_1 and Δ_2 of the excitations vanishing for $g = 1$ in both cases.
- Furthermore, both dispersion relations can be transformed into one-another by making the substitution $g \leftrightarrow 1/g$ (up to a prefactor in the energy).

Therefore, both domain walls (in the high coupling limit $g \ll 1$) and flipped-spin excitations (in the low coupling limit $g \gg 1$) seem to describe similar quasi-particle physics of the excitations in their respective validity regime. While they are different excitations, this hints at the self-duality found in the Task 2.
For $J_2 \neq 0$ the energy gap of the domain-wall excitations increases. This is simply because it becomes more costly for next-nearest neighbors to have opposite orientation. In the case of the flipped-spin dispersion relation, the effective mass of the excitations decreases more than one would expect from the duality observed for $J_2 = 0$. Namely, $(2m_2)^{-1} = J + 4J_2$, while Δ_1 would suggest that $(2\tilde{m}_2)^{-1} = J + 2J_2$. This is because in the case of a single flipped-spin excitation the additional hopping term $J_2\sigma_n^z\sigma_{n+2}^z$ acts on both sides of the excitation, while the energy penalty for the domain-wall is only incurred on one. Hence, the spin hops faster (is lighter) than expected from the simple duality argument.

21. Self-duality of the quantum Ising chain

We wish to derive the dual representation of the one-dimensional quantum Ising chain in a transverse field,

$$H_{\mathrm{I}} = -J \sum_n (\sigma_n^z \sigma_{n+1}^z + g\sigma_n^x). \tag{32}$$

(a) First, introduce spin operators on the dual lattice, i.e., the lattice where the sites are given by the bonds of the original lattice,

$$\begin{aligned} \tau_n^x &= \sigma_{n+1}^z \sigma_n^z, \\ \tau_n^z &= \prod_{m \leq n} \sigma_m^x, \end{aligned} \tag{33}$$

and show that they satisfy the algebra in Eqs. 31 as well.

First, we notice that

$$\tau_n^x \tau_n^x = \sigma_{n+1}^z \sigma_n^z \sigma_{n+1}^z \sigma_n^z = 1, \qquad \tau_n^z \tau_n^z = \prod_{m \leq n} \prod_{m' \leq n} \sigma_m^x \sigma_{m'}^x = 1$$

giving the first part of the algebra. Second, we compute the onsite commutator term finding

$$\tau_n^z \tau_n^x = \prod_{m \leq n} \sigma_m^x \sigma_{n+1}^z \sigma_n^z = \sigma_{n+1}^z \sigma_n^x \sigma_n^z \prod_{m \leq n-1} \sigma_m^x = -\sigma_{n+1}^z \sigma_n^z \prod_{m \leq n} \sigma_m^x = -\tau_n^x \tau_n^z.$$

Lastly, to obtain the third condition, consider $n \geq m+1$, such that

$$\tau_n^z \tau_m^x = \prod_{k \leq n} \sigma_k^x \sigma_{m+1}^z \sigma_m^z = \sigma_{m+1}^x \sigma_{m+1}^z \sigma_m^x \sigma_m^z \prod_{i > m+1}^{n} \sigma_i^x \prod_{i < m} \sigma_i^x = (-1)^2 \tau_m^x \tau_n^z = \tau_m^x \tau_n^z$$

and for $n \leq m-1$

$$\tau_n^z \tau_m^x = \prod_{i \leq n} \sigma_i^x \sigma_{m+1}^z \sigma_m^z = \sigma_{m+1}^z \sigma_m^z \prod_{i \leq n} \sigma_i^x = \tau_m^x \tau_n^z.$$

Therefore, the τ_n^z and τ_n^x satisfyy the same algebra as the σ_n^z and σ_n^x in Eqs. 31.

(b) Express the Hamiltonian in Eq. 32 in terms of the dual operators.

Note, that

$$\tau_n^z \tau_{n+1}^z = \prod_{m \leq n} \sigma_m^x \prod_{m' \leq n+1} \sigma_{m'}^x = \sigma_{n+1}^x.$$

Therefore,

$$H_{\mathrm{I}} = -J\sum_n(\sigma_n^z\sigma_{n+1}^z + g\sigma_n^x) \to \tilde{H}_{\mathrm{I}} = -Jg\sum_n(\tau_n^z\tau_{n+1}^z + g^{-1}\tau_n^x).$$

(c) Use the dual Hamiltonian to derive a relation between the energy eigenvalues at coupling g and coupling $1/g$.

By the similarity of the Ising Hamiltonian and its dual, and the fact that the τ and σ abide the same algebra, we can conclude that the energy eigenvalues $E(J,g)$ map to the eigenvalues $\tilde{E}(Jg,1/g)$ in the dual problem.

(d) Argue that the critical point of H_{I} is at $g = 1$. Which further assumptions are needed?

If we assume that the energies are continuous functions of the coupling parameters, then the critical point must occur at $g = 1$. There, the two energies are equal and therefore the domain-wall excitations and flipped-spin excitations become in some sense equal.

22. Shift exponent in the quantum ϕ^4 theory

The ϕ^4 field theory with the action

$$S = \int \mathrm{d}^d x \mathrm{d}\tau \left[\frac{1}{2}\left(c^2(\nabla\phi_a)^2 + (\partial_\tau\phi_a)^2 + r_0\phi_a^2\right) + \frac{u_0}{4!}(\phi_a^2)^2\right] \tag{34}$$

with $a = 1, 2, \ldots, N$, has a quantum phase transition at $T = 0$, $r_0 = r_{\mathrm{c}}$. The shift exponent ψ is defined via the temperature-dependent phase boundary

$$T_{\mathrm{c}} \sim (r_{\mathrm{c}} - r_0)^{\psi}, \tag{35}$$

where T_{c} is the critical temperature. To calculate T_{c}, note that the phase transition occurs when the renormalized temperature-dependent mass $r(T)$ of the order parameter vanishes. The upper critical dimension for the quantum phase transition is $d_{\mathrm{c}}^+ = 4 - z = 3$.

(a) Below the upper critical dimension d_{c}^+, use a simple scaling argument to relate ψ to other critical exponents.

The key idea is that the finite temperature $T \approx T_{\mathrm{c}} > 0$ introduces a length scale ξ_T which competes with the correlation length. We consider a small temperature difference δT from the critical point, i.e., $T = T_{\mathrm{c}} + \delta$.
The thermal energy scale is proportional to $E_{\mathrm{th}} = k_B T$ and the correlation time $E_{\mathrm{ct}} = \hbar\tau^{-1}$. Now, $E_{\mathrm{th}} \sim E_{\mathrm{ct}}$ yields

$$T \propto \tau^{-1} \propto \xi_T^{-z}.$$

As $T \propto T_\mathrm{c}$, we have $\xi_T \propto T^{-1/z}$.
Using the correlation length scaling, we find

$$\xi \propto |t|^{-\nu} \propto (r_0 - r_\mathrm{c})^{-\nu}.$$

These two quantities compete for $\xi \sim \xi_T$, yielding

$$\xi_T \propto T_\mathrm{c}^{-1/z} \sim \xi \propto (r_0 - r_\mathrm{c})^{-\nu}.$$

We find $T_\mathrm{c} \propto (r_0 - r_\mathrm{c})^{\nu z}$ from which we identify $\nu z = \psi$.

(b) For $d > d_\mathrm{c}^+$, the naive scaling analysis above becomes invalid. However, a perturbative calculation of $r(T)$ becomes feasible. To this end, calculate the self-energy of the ϕ propagator in bare perturbation theory to first order in u_0. The temperature dependence of $r(T)$ at $r_0 = r_\mathrm{c}$ allows to obtain ψ in this case.

For finite T we perform the Fourier transform

$$\phi_a(\vec{x}, \tau) = \frac{1}{\beta} \int \frac{\mathrm{d}^d \vec{k}}{(2\pi)^d} \sum_{\omega_n} \phi_a(\vec{k}, \omega_n) e^{i\vec{k}\vec{x} + i\omega_n \tau},$$

where $\omega_n = \beta^{-1} 2\pi n$, $n \in \mathbb{Z}$ are the bosonic Matsubara frequencies, yielding for the bare propagator in our theory

$$G_0 = -\beta(c^2 \vec{q}^2 + \omega^2 + r_0).$$

Hence, the first interaction correction to the mass term r_0 is given by the Feynman diagram

$$\delta r(T) \propto \underset{u_0}{\text{---}\!\bigcirc\!\text{---}} \propto u_0 \int \frac{\mathrm{d}^d \vec{q}}{(2\pi)^d} T \sum_n \frac{1}{c^2 \vec{q}^2 + (2\pi n T)^2 + r_0}.$$

The crucial observation is then, that one does not need to evaluate this integral exactly to obain the scaling behavior. Instead, by substituting $\vec{q} = T\tilde{\vec{q}}$ and $\mathrm{d}^d \vec{q} = T^d \mathrm{d}^d \tilde{\vec{q}}$ we can write the integral as

$$\delta r(T) \propto T^{d-1} u_0 f(r_0 / T^2),$$

having defined the function

$$f(x) = \int \frac{\mathrm{d}^d \vec{q}}{(2\pi)^d} \sum_n \frac{1}{c^2 \vec{q}^2 + (2\pi n)^2 + x}.$$

Therefore, close to the critical point

$$T_\mathrm{c} \propto \delta r^{\frac{1}{d-1}}$$

and hence $\psi = 1/(d-1)$.

(c) Apply the procedure of (b) to a situation with $z = 2$ where the bare propagator is $G^{-1} = i\omega_n - c^2\vec{k}^2 - r_0$ (instead of $G^{-1} = -\omega_n^2 - c^2\vec{k}^2 - r_0$).

We apply the same procedure as above but replace the propagator to obtain the self-energy correction in first order as

$$\delta r(T) \propto \underset{u_0}{\bigcirc\!\!-\!\!-} \propto u_0 \int \frac{\mathrm{d}^d\vec{q}}{(2\pi)^d} T \sum_n \frac{1}{c^2\vec{q}^2 - i2\pi nT + r_0}.$$

We use a similar trick, but substitute $\vec{q} = \sqrt{T}\tilde{\vec{q}}$ and $\mathrm{d}^d\vec{q} = T^{d/2}\mathrm{d}^d\tilde{\vec{q}}$ instead, obtaining

$$\delta r(T) \propto T^{\frac{d}{2}} u_0 g(r_0/T)$$

with function

$$g(x) = \int \frac{\mathrm{d}^d\vec{q}}{(2\pi)^d} \sum_n \frac{1}{c^2\vec{q}^2 - i2\pi n + x}.$$

Therefore, $\psi = 2/d$.

23. Quantum critical point in the dilute Bose gas

Consider the quantum critical point in the dilute Bose gas with the action

$$S = \int \mathrm{d}^d x \mathrm{d}\tau (\Phi^* \partial_\tau \Phi + v \left|\partial_\tau \Phi\right|^2 + \left|\nabla\Phi\right|^2 - \mu \left|\Phi\right|^2 + \lambda \left|\Phi\right|^4) \tag{36}$$

in $d = 2 - \epsilon$ dimensions.

(a) What is the scaling dimension of v?

We perform the rescaling $x \mapsto bx$, $\tau \mapsto b^z\tau$ and $\Phi \mapsto b^y\Phi$, such that the rescaled action reads

$$\begin{aligned} S = \int b^{d+z} \mathrm{d}^d x \mathrm{d}\tau (&\Phi^* b^{2y-z} \partial_\tau \Phi + v b^{2y-2z} \left|\partial_\tau \Phi\right|^2 + b^{2y-2} \left|\nabla\Phi\right|^2 \\ &- b^{2y}\mu \left|\Phi\right|^2 + b^{4y}\lambda \left|\Phi\right|^4). \end{aligned}$$

To temporal term invariant, we require

$$d + z + 2y - z = 0 \implies y = \frac{d}{2}.$$

Therefore, the coupling v scales as

$$v \mapsto b^{d+z+2y-2z} v = b^{-z} v.$$

Hence, the scaling dimension is $-z$, which renders this coupling irrelevant.
This may also be obtained by power-counting with $[\mathrm{d}x] = -1$, $[\nabla] = 1$, $[\mathrm{d}\tau] = -z$ and $[\partial_\tau] = z$, yielding for the dimension of the field $[\Phi]$ in the temporal term

$$0 = [S] = d[\mathrm{d}x] + [\mathrm{d}\tau] + [\partial_\tau] + 2[\Phi] \implies [\Phi] = \frac{d}{2}.$$

Then, for the term with the coupling v, we find

$$0 = [S] = d[\mathrm{d}x] + [\mathrm{d}\tau] + [v] + 2[\partial_\tau] + 2[\Phi] \implies [v] = -z.$$

(b) Show that the RG flow of the quartic selfinteraction λ is given by

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\ln b} = \epsilon\lambda - \lambda^2 \tag{37}$$

with suitably rescaled dimensionless λ.

The linear term can already be obtained from the scaling arguments above. For this, note that the kinetic term scales with $b^{d+z+2y-2} = b^{z-2}$. By choosing $z = 2$, we make this term scale invariant. Then, for the coupling λ, we have

$$\lambda \mapsto b^{d+z+4y}\lambda = b^{z-d}\lambda = b^\epsilon\lambda,$$

for $z = 2$ and $d = 2 - \epsilon$, such that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\ln b} = \epsilon\lambda$$

to first order in λ. Furthermore, we know that the coupling v is irrelevant and ignore it from now on.
To this end, note that the chemical potential term μ scales as $b^{d+z+2y} = b^z$, hence the RG flow for this term is

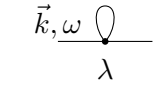
$$\frac{\mathrm{d}\mu}{\mathrm{d}\ln b} = z\mu = 2\mu.$$

To compute the higher order corrections, we need to consider the renormalized action in momentum space given by

$$\begin{aligned} S_{\text{eff}} = &\int_0^{\Lambda/b} \frac{\mathrm{d}^d\vec{q}}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} (Z_\omega \Phi^* i\omega\Phi + Z_\Phi \Phi^* \vec{q}^2 \Phi - Z_\mu \mu \Phi^*\Phi) \\ &+ \int_0^{\Lambda/b} \frac{\mathrm{d}^d\vec{q}_1 \mathrm{d}^d\vec{q}_2 \mathrm{d}^d\vec{q}_3}{(2\pi)^{3d}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_3}{(2\pi)^3} Z_\lambda \lambda \Phi^*\Phi\Phi^*\Phi, \end{aligned}$$

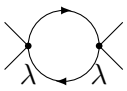
where Z_x are the respective constants obtained by integrating out the high-energy modes.
The first-order interaction corrections to the coefficients can be computed from

the following Feynman diagrams



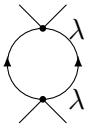
$$= (-1)(-\lambda)(\overbrace{\Phi^*\Phi}\Phi^*\Phi) \times 2^2$$
$$= 4\lambda(\Phi^*\Phi)\int_{\Lambda/b}\frac{\mathrm{d}^d\vec{q}}{(2\pi)^d}\underbrace{\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{1}{+i\omega+\vec{q}^2-\mu}}_{=0} = 0,$$

where in the first term, -1 comes from reexponentiation and 2^2 are the ways to choose the contracted Φ (Φ^*). The integral is zero, because its only pole is at $i(q^2-\mu)$ and hence for suitable regularization this integral vanishes.
The second-order interaction corrections are similarly



$$= (-1)\frac{(-\lambda)^2}{2!}(\Phi^*\Phi\overbrace{\Phi^*\Phi)(\Phi^*}\Phi\Phi^*\Phi) \times 2^2 \times 2^2$$
$$= -8\lambda^2(\Phi^*\Phi\Phi^*\Phi)\int_{\Lambda/b}\frac{\mathrm{d}^d\vec{q}}{(2\pi)^d}\underbrace{\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{1}{(i\omega+\vec{q}^2-\mu)(i\omega+\vec{q}^2-\mu)}}_{=0},$$

because all poles are in the same complex plane and



$$= (-1)\frac{(-\lambda)^2}{2!}(\Phi^*\overbrace{\Phi\Phi^*\Phi)(\Phi^*}\Phi\Phi^*\Phi) \times 2^2$$
$$= -2\lambda^2(\Phi^*\Phi\Phi^*\Phi)\int_{\Lambda/b}\frac{\mathrm{d}^d\vec{q}}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\underbrace{\frac{1}{(i\omega+\vec{q}^2-\mu)(-i\omega+\vec{q}^2-\mu)}}_{=\frac{1}{\omega^2+(q^2-\mu)^2}}$$
$$= -2\lambda^2(\Phi^*\Phi\Phi^*\Phi)\int_{\Lambda/b}\frac{\mathrm{d}^d\vec{q}}{(2\pi)^d}\frac{1}{2q^2} + \mathcal{O}(\mu)$$
$$= -\frac{S_d}{(2\pi)^d}\Lambda^{d-2}\ln b\lambda^2(\Phi^*\Phi\Phi^*\Phi) + \mathcal{O}(\mu, \ln b).$$

The last term provides the correction to the coupling λ. By defining $\tilde{\mu} \equiv \mu/\Lambda^2$ and $\tilde{\lambda} \equiv (S_d/(2\pi)^d)\lambda/\Lambda^{2-d}$, we hence obtain the RG flow equations

$$\frac{\mathrm{d}\tilde{\mu}}{\mathrm{d}\ln b} = 2\tilde{\mu}, \qquad \frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\ln b} = \epsilon\tilde{\lambda} - \tilde{\lambda}^2.$$

(c) Determine the critical exponents ν, η and z to the leading order in ϵ for $d < 2$.

We already determined $z = 2$ in part (b). For ν, note that the mass term gets rescaled as $\mu \to b^2\mu$ and therefore $\nu = \frac{1}{2}$.
For the anomalous dimension η, we consider the self-energy up to first order in λ, for which we found

$$\Sigma(\vec{k}) = \quad \vec{k},\omega \quad \lambda \quad = 0.$$

Hence, $\eta = 0$.