Exercises for "Quantum Phase Transitions" **Summer 24**

Dr. L. Janssen **Exercise 6 (12.07.24)**

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1. Quantum Ising chain with second-neighbor exchange (4 points)

Consider a quantum Ising chain with second-neighbor exchange in a transverse field,

$$H_{\mathrm{I}} = -\sum_n \left(J\sigma_n^z \sigma_{n+1}^z + J_2 \sigma_n^z \sigma_{n+2}^z + Jg\sigma_n^x \right). \tag{1}$$

Here, the spin-1/2 operators are represented by Pauli matrices σ_n^x and σ_n^y that fulfill the algebra:

$$\begin{aligned} \sigma_n^z \sigma_n^z &= \sigma_n^x \sigma_n^x = 1, \\ \sigma_n^z \sigma_n^x &= -\sigma_n^x \sigma_n^z, \\ \sigma_n^z \sigma_m^x &= \sigma_m^x \sigma_n^z, \qquad \text{for } m \neq n. \end{aligned} \tag{2}$$

(a) Determine the dispersion relation of a domain-wall excitation to lowest order in g.

(b) Determine the dispersion relation of a flipped-spin excitation in the limit $g \gg 1$.

(c) Interpret the results.

2. Self-duality of the quantum Ising chain (6 points)

We wish to derive the dual representation of the one-dimensional quantum Ising chain in a transverse field,

$$H_{\mathrm{I}} = -J \sum_n \left(\sigma_n^z \sigma_{n+1}^z + g\sigma_n^x \right). \tag{3}$$

(a) First, introduce spin operators on the dual lattice, i.e., the lattice where the sites are given by the bonds of the original lattice,

$$\begin{aligned} \tau_n^x &= \sigma_{n+1}^z \sigma_n^z, \\ \tau_n^z &= \prod_{m \leq n} \sigma_m^x, \end{aligned} \tag{4}$$

and show that they satisfy the algebra (2) as well.

(b) Express the Hamiltonian (3) in terms of the dual operators.

(c) Use the dual Hamiltonian to derive a relation between the energy eigenvalues at coupling g and coupling $1/g$.

(d) Argue that the critical point of H_{I} is at $g = 1$. Which further assumptions are needed?