

3.1 Coherent-state path integral

Prototypical example: system of (many!) interacting bosons

Fock basis of many-body Hilbert space:

$$|n_1, n_2, \dots, n_N\rangle = \prod_{\alpha=1}^N \frac{(\hat{a}_{\alpha}^+)^{n_{\alpha}}}{\sqrt{n_{\alpha}!}} |0\rangle$$

where $|0\rangle = |0, \dots, 0\rangle$ is the (many-body) vacuum state and \hat{a}_{α}^+ (\hat{a}_{α}) creates (annihilates) a boson in the α -th single-particle state.

Commutation relation:

$$[\hat{a}_{\alpha}, \hat{a}_{\alpha'}^+] = \delta_{\alpha, \alpha'} \quad \text{and} \quad [\hat{a}_{\alpha}, \hat{a}_{\alpha'}] = [\hat{a}_{\alpha}^+, \hat{a}_{\alpha'}^+] = 0$$

Expansion in basis states:

$$|\Phi\rangle = \sum_{n_1=0}^{\infty} \dots \sum_{n_N=0}^{\infty} \Phi_{n_1, \dots, n_N} |n_1, \dots, n_N\rangle$$

Coherent state:

$$\hat{a}_{\alpha} |\Phi\rangle = \Phi_{\alpha} |\Phi\rangle \quad \text{for all } \alpha=1, \dots, N$$

where $\Phi_{\alpha} \in \mathbb{C}$.

Expansion coefficients:

$$\Phi_{n_1, \dots, n_N} = \prod_{\alpha=1}^N \frac{(\Phi_{\alpha})^{n_{\alpha}}}{\sqrt{n_{\alpha}!}}$$

Then:

$$|\Phi\rangle = \sum_{n_1=0}^{\infty} \dots \sum_{n_N=0}^{\infty} \prod_{\alpha=1}^N \frac{(\Phi_{\alpha} \hat{a}_{\alpha}^+)^{n_{\alpha}}}{n_{\alpha}!} |0\rangle = e^{\sum_{\alpha} \Phi_{\alpha} \hat{a}_{\alpha}^+} |0\rangle \quad (\text{not normalized})$$

Remarks:

- $|\Phi\rangle$ as defined above can easily be shown to satisfy $\hat{a}_{\alpha} |\Phi\rangle = \Phi_{\alpha} |\Phi\rangle$ by using $[\hat{a}_{\alpha}, (\hat{a}_{\alpha'}^+)^n] = n(\hat{a}_{\alpha'}^+)^{n-1} \delta_{\alpha\alpha'}$
- $|\Phi\rangle$ is a superposition of states with arbitrary number of particles

Bra version:

$$\langle\Phi| = \langle 0| e^{\sum_{\alpha} \Phi_{\alpha}^* \hat{a}_{\alpha}} \quad \text{with} \quad \langle\Phi| \hat{a}_{\alpha}^+ = \langle\Phi| \Phi_{\alpha}^*$$

Particle creation:

$$\hat{a}_{\alpha}^+ |\Phi\rangle = \frac{\partial}{\partial \Phi_{\alpha}} |\Phi\rangle$$

Overlap of two coherent states:

$$\langle\Phi|\Phi'\rangle = e^{\sum_{\alpha} \Phi_{\alpha}^* \Phi_{\alpha'}} \neq 0 \quad (\text{not orthogonal})$$

Resolution of unity:

$$\mathbb{1} = \int \prod_{\alpha} \frac{d\Phi_{\alpha}^* d\Phi_{\alpha}}{2\pi i} e^{-\sum_{\alpha} \Phi_{\alpha}^* \Phi_{\alpha}} |\Phi\rangle \langle\Phi|$$

\Rightarrow Coherent states form an overcomplete "basis"

NB: $\int \frac{d\phi^* d\phi}{2\pi i} e^{-|\phi|^2} |\phi\rangle \langle\phi|$

$$= \int_0^{\infty} \frac{r dr}{\pi} \int_0^{2\pi} d\theta e^{-r^2} \sum_{m,n=0}^{\infty} \frac{(r e^{i\theta})^m}{\sqrt{m!}} |m\rangle \frac{(r e^{-i\theta})^n}{\sqrt{n!}} \langle n|$$

$$= \sum_{h=0}^{\infty} \frac{1}{h!} \int_0^{\infty} dr 2r e^{-r^2} r^{2h} |h\rangle \langle h| = \sum_h |h\rangle \langle h| = \mathbb{1}$$

Grand canonical partition function :

$$Z = \text{Tr} e^{-\beta(\hat{H} - \mu \hat{N})}$$

where $\beta = \frac{1}{k_B T}$ the inverse temperature and μ the chemical potential

Hamiltonian (in "second quantization") :

$$\hat{H} = \sum_{\alpha} e_{\alpha} \hat{a}_{\alpha}^{\dagger} \hat{a}_{\alpha} + \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \beta | V | \gamma \delta \rangle \hat{a}_{\alpha}^{\dagger} \hat{a}_{\beta}^{\dagger} \hat{a}_{\gamma} \hat{a}_{\delta}$$

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↑
 energy eigenvalues two-body
 for $V=0$ interaction

Particle number operator :

$$\hat{N} = \sum_{\alpha} \hat{a}_{\alpha}^{\dagger} \hat{a}_{\alpha}$$

Partition function :

$$Z = \int \prod_{\alpha} \frac{d\bar{\Phi}_{\alpha}^* d\Phi_{\alpha}}{2\pi i} e^{-\sum_{\alpha} \bar{\Phi}_{\alpha}^* \Phi_{\alpha}} \langle \bar{\Phi} | e^{-\beta(\hat{H} - \mu \hat{N})} | \Phi \rangle$$

$\beta = \epsilon M$
 $(M \gg 1)$

$e^{-\epsilon(\hat{H} - \mu \hat{N})} = e^{-\epsilon(\hat{H} - \mu \hat{N})} \dots e^{-\epsilon(\hat{H} - \mu \hat{N})}$
 $\mathbb{1} = \int \prod \frac{d\bar{\Phi}^* d\Phi}{2\pi i} e^{-\sum \bar{\Phi}^* \Phi} |\Phi\rangle \langle \bar{\Phi}|$

$$= \int \prod_{k=0}^{M-1} \prod_{\alpha} \frac{d\bar{\Phi}_{\alpha, k}^* d\Phi_{\alpha, k}}{2\pi i} e^{-\sum_{k=0}^{M-1} \sum_{\alpha} \bar{\Phi}_{\alpha, k}^* \Phi_{\alpha, k}} \prod_{k=0}^{M-1} \langle \bar{\Phi}_k | e^{-\epsilon(\hat{H} - \mu \hat{N})} | \Phi_{k+1} \rangle$$

$\langle \bar{\Phi}_k | \Phi_{k+1} \rangle = \epsilon \langle \bar{\Phi}_k | (\hat{H} - \mu \hat{N}) | \Phi_{k+1} \rangle + O(\epsilon^2)$

where $\bar{\Phi}_0 = \bar{\Phi}_M = \bar{\Phi}$

Normal ordered operators: all \hat{a}_α^+ are left of all \hat{a}_α

$$\langle \Phi | A(\hat{a}_\alpha^+, \hat{a}_\alpha) | \Phi' \rangle = A(\Phi_\alpha^*, \Phi'_\alpha) e^{\sum_\alpha \Phi_\alpha^* \Phi'_\alpha}$$

↑
arbitrary normal-ordered function of $\hat{a}_\alpha^+, \hat{a}_\alpha$

since $a_\alpha | \Phi' \rangle = \Phi'_\alpha | \phi \rangle$ and $\langle \Phi | a_\alpha^+ = \langle \Phi | \Phi_\alpha$

NB: $\tilde{H} = \hat{H} - \mu \hat{N}$

$$= \langle \phi_k | \phi_{k+1} \rangle - \epsilon \langle \phi_k | \tilde{H} | \phi_{k+1} \rangle$$

$$= e^{\sum_k \phi_k^* \phi_{k+1}} \underbrace{(1 - \epsilon \tilde{H}(\phi_k^*, \phi_{k+1}))}_{e^{-\epsilon \tilde{H}(\phi_k^*, \phi_{k+1})}}$$

Then:

$$Z = \lim_{M \rightarrow \infty} \int \prod_{k=0}^{M-1} \prod_\alpha \frac{d\phi_{\alpha,k}^* d\phi_{\alpha,k}}{2\pi i} e^{-\sum_{k=0}^{M-1} \sum_\alpha \Phi_{\alpha,k}^* (\Phi_{\alpha,k} - \Phi_{\alpha,k+1})}$$

$$\times e^{-\sum_{k=0}^{M-1} \epsilon [H(\Phi_{\alpha,k}^*, \Phi_{\alpha,k+1}) - \mu \sum_\alpha \Phi_{\alpha,k}^* \Phi_{\alpha,k+1}]}$$

$$=: \int_{\Phi_\alpha(0) = \Phi_\alpha(\beta)} \mathcal{D}\Phi_\alpha^*(\tau) \mathcal{D}\Phi_\alpha(\tau) e^{-S[\Phi_\alpha^*(\tau), \Phi_\alpha(\tau)]}$$

"functional integral"

with the "imaginary time" $\tau = \frac{k}{M} \beta$ and the action:

$$S = \int_0^\beta d\tau \left\{ \sum_\alpha \Phi_\alpha^*(\tau) (-\partial_\tau - \mu) \Phi_\alpha(\tau) + H[\Phi_\alpha^*(\tau), \Phi_\alpha(\tau)] \right\}$$

Remarks:

- $\mathcal{D}\Phi_\alpha^*$ and $\mathcal{D}\Phi_\alpha$ should be understood as the "sum" over all complex functions $\Phi_\alpha(\tau)$ that satisfy $\Phi_\alpha(0) = \Phi_\alpha(\beta)$
- Quantum number α labels states in the single-particle basis, e.g., momentum \vec{q} , position \vec{x} , lattice site i

Example (nonrel. bosons of mass m interacting via $V(\vec{x}-\vec{y})$):

$$S = \int_0^\beta d\tau \int d^d\vec{x} \left[\Phi^*(\vec{x}, \tau) \left(-\partial_\tau^2 - \mu - \frac{\hbar^2 \vec{\nabla}^2}{2m} \right) \Phi(\vec{x}, \tau) + \int d^d\vec{y} |\Phi(\vec{x}, \tau)|^2 V(\vec{x}-\vec{y}) |\Phi(\vec{y}, \tau)|^2 \right]$$

Example ("Higgs" bosons interacting via $V(\vec{x}-\vec{y}) = \lambda \delta(\vec{x}-\vec{y})$):

$$S = \int_0^\beta d\tau \int d^d\vec{x} \left[\frac{1}{2} \Phi(\vec{x}, \tau) \left(-\partial_\mu^2 + m^2 \right) \Phi(\vec{x}, \tau) + \lambda \Phi(\vec{x}, \tau)^4 \right],$$

↑ "Higgs" mass

where $(\partial_\mu) = (\frac{1}{c} \frac{\partial}{\partial \tau}, \vec{\nabla})$, $\mu = 0, \dots, d$.

Example (Dirac fermion coupled to order-parameter field):

$$S = \int_0^\beta d\tau \int d^d\vec{x} \left[\overset{\text{e.g., quarks}}{\Psi(\vec{x}, \tau)} \gamma^\mu \partial_\mu \overset{\text{e.g., mesons}}{\Psi(\vec{x}, \tau)} + \frac{1}{2} \Phi(\vec{x}, \tau) \left(-\partial_\mu^2 + m^2 \right) \Phi(\vec{x}, \tau) + g \Phi(\vec{x}, \tau) \bar{\Psi}(\vec{x}, \tau) \Psi(\vec{x}, \tau) \right]$$

↑ Yukawa coupling

where $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} \mathbb{1}$ Dirac matrices

Fourier transform:

(30)

$$\Phi(\vec{r}, \tau) = \frac{1}{\beta} \int \frac{d\vec{k}}{(2\pi)^d} \sum_{\omega_n = \frac{2\pi n}{\beta}} \Phi(\vec{k}, \omega_n) e^{i\vec{k}\cdot\vec{r} + i\omega_n \tau}, \quad n=0, \pm 1, \pm 2, \dots$$

↑
Matsubara
frequencies

Action (nonrel. bosons with $V=0$):

$$S = \frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \sum_{\omega_n} \left(-i\omega_n + \frac{\hbar^2 k^2}{2m} - \mu \right) |\Phi(\vec{k}, \omega_n)|^2$$

Propagator:

$$G(\omega_n, k) = \frac{\beta}{-i\omega_n + \frac{\hbar^2 k^2}{2m} - \mu} = \beta \frac{+i\omega_n + \frac{\hbar^2 k^2}{2m} - \mu}{\omega_n^2 + \left(\frac{\hbar^2 k^2}{2m} - \mu \right)^2}$$

Remarks:

- For $T > 0$ all modes with $\omega_n \neq 0$ ($|n| \geq 1$) have a finite gap $\propto T$
 \Rightarrow cannot contribute to a nonanalyticity in $Z = \int \mathcal{D}\Phi \mathcal{D}\Phi^* e^{-S}$,
 \Rightarrow "non-critical modes"
- For $T > 0$ system can be understood as having finite extent β in imaginary time $\tau \in [0, \beta)$
 \Rightarrow correlation time τ_c at criticality: $\tau_c \gg \beta$
 \Rightarrow critical configurations $\Phi(\tau) \approx \Phi(0)$ for all $\tau \in [0, \beta)$.
 \Rightarrow only $\omega_{n=0} = 0$ mode contributes to critical properties at finite T
- Statics and dynamics decouple at $T > 0$

• For $T=0$ we have $\tau \in (0, \infty)$ and ω becomes

continuous:

$$\frac{1}{\beta} \sum_{\omega_n} \rightarrow \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}$$

\Rightarrow continuum of small- ω modes contribute at a quantum critical point

\Rightarrow quantum critical behavior in d dimensions (often) resembles classical critical behavior in $d+z$ dimensions ("quantum-to-classical mapping", see Chapter 6)

3.2 Mean-field approximation: Landau theory

Assume critical point at $T > 0$: retain only critical ($\omega_0=0$) modes

Effective action (bosons with $V(\vec{x}-\vec{y}) = \lambda \delta(\vec{x}-\vec{y})$):

$$S[\Phi] = \beta \int d^d \vec{\tau} \left[\frac{\hbar^2}{2m} |\vec{\nabla} \Phi(\vec{\tau})|^2 - \mu |\Phi(\vec{\tau})|^2 + \lambda |\Phi(\vec{\tau})|^4 \right]$$

Remark:

Using $\Phi = \phi_1 + i\phi_2$ with $\phi_a, a=1,2$ real scalar field, the action reads

$$S[\vec{\phi}] = \beta \int d^d \vec{\tau} \left[\sum_{a=1}^N \frac{\hbar^2}{2m} (\vec{\nabla} \phi^a)^2 - \sum_{a=1}^N \mu \phi^a \phi^a + \lambda \left(\sum_{a=1}^N \phi^a \phi^a \right)^2 \right]$$

for $N=2$.

"Ginzburg-Landau-Wilson theory"

" $O(N)$ model"

" ϕ^4 theory"

Partition function:

$$Z = \int \mathcal{D}\vec{\phi} e^{-S[\vec{\phi}]}$$

will be dominated by configurations that minimize $S[\vec{\phi}]$

Saddle-point approximation (= mean-field approximation)

$$Z \approx e^{-S[\vec{\phi}_0]} \quad \text{with} \quad \left. \frac{\delta S}{\delta \vec{\phi}} \right|_{\vec{\phi}_0} = 0 \quad \text{and} \quad \left. \frac{\delta^2 S}{\delta \vec{\phi} \delta \vec{\phi}^T} \right|_{\vec{\phi}_0} \text{ pos. definite}$$

where we have neglected fluctuations $S[\phi] - S[\phi_0] = \frac{1}{2} \delta \vec{\phi}^T \cdot \left. \frac{\delta^2 S}{\delta \vec{\phi} \delta \vec{\phi}^T} \right|_{\vec{\phi}_0} \delta \vec{\phi} + \mathcal{O}(\delta \phi^3)$

Free energy:

$$F = -k_B T \ln Z$$

$$= \underline{\underline{k_B T S[\vec{\phi}_0]}}$$

\Rightarrow recovers Landau-Ginzburg theory with parameters

$$(a, b, \frac{c_0^2}{\xi_0}) \propto (-2\mu, 4\lambda, \frac{\hbar^2}{2m})$$