4 Renormalization group (RG)

4.1 Concept of RG

Assumption: The relevant physics describing phases and phase transition is governed by the behavior at large length scales ($\hat{=}$ low energy) $\underset{\substack{\uparrow \\ \text{typical length} \\ \text{scale}}}{L} \gg \underset{\nwarrow \text{microscopic length scale}}{a}$

Example (magnet): $\langle S^z_i \, S^z_{i+1} \rangle \neq 0$ for all T

$$\lim_{|i-j| \to \infty} \langle S^z_i S^z_j \rangle \begin{cases} = 0 & \text{for } T > T_c \\ \neq 0 & \text{for } T < T_c \end{cases}$$

Idea: Successively integrate out short-distance ($\hat{=}$ high-energy) modes to obtain an <u>effective theory</u> at large length scales ($\hat{=}$ low energy)

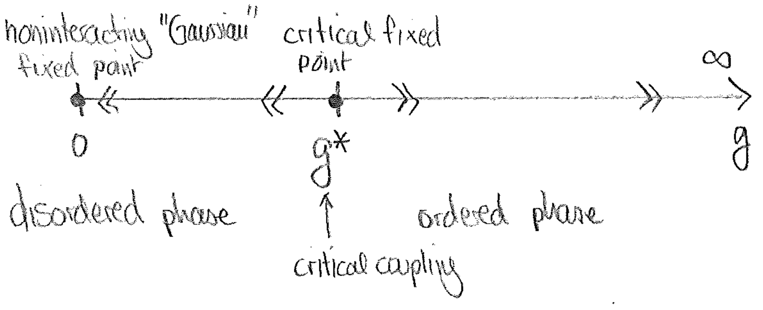
E.g.: $\underset{\nearrow \text{action}}{S}(\underset{\substack{\uparrow \\ \text{couplings}}}{g_1, g_2, \dots}) \longmapsto \underset{\nwarrow \text{if } S \text{ is sufficiently general}}{S}(g_1', g_2', \dots)$

$\Rightarrow$ couplings become <u>scale-dependent</u> $g_i \to g_i(L)$

RG flow:

Change of couplings under the successive integration of modes

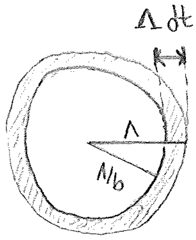
RG flow diagram (example):



4.2 Scaling transformation and scaling dimension

Integrate out high-energy modes with momenta:

$$\frac{\Lambda}{b} \leq |\vec{k}| < \Lambda$$



"momentum shell"

Infinitesimal RG step:

$$b = e^{dt} \quad \text{with} \quad dt \ll 1 \quad , \quad t = \int_0^t dt' \quad \text{"RG time"}$$

RG flow:

$$\frac{dg_i}{dt} = \beta_i(g_j)$$

↑ beta functions

Fixed point:

$$\left.\frac{dg_i}{dt}\right|_{g_i^*} = 0$$

Linearized RG flow (around Gaussian fixed point $g^* = 0$ in theory with one coupling g):

$$\beta(g) = \theta\, g + \mathcal{O}(g^2) \quad \text{with } \theta \equiv \dim[g] = \text{const.} \quad \text{"}\underline{\text{scaling dimension}}\text{ of } g\text{"}$$

↑ constant term vanishes since $g=0$ is a fixed point

Integrated flow:

$$\frac{dg}{dt} = \theta\, g \quad \Rightarrow \quad g(t) = g(0)\, e^{\theta t}$$

Classification of couplings:

$\dim[g] > 0$	"<u>relevant</u> coupling"	g increases in RG time
$\dim[g] < 0$	"<u>irrelevant</u> coupling"	g decreases in RG time
$\dim[g] = 0$	"<u>marginal</u> coupling"	higher-order terms decide its fate (marginally relevant, marginally irrelevant, or exactly marginal)

Classification of fixed points:

"stable fixed point": all couplings irrelevant near fixed point

"critical fixed point": exactly one relevant direction

"multicritical fixed point": number of relevant directions $2 \leq n \leq n_0$
where n_0 is the number of tuning parameters

"unstable fixed point": number of relevant directions $n > n_0$

4.3 Momentum-shell RG for the O(N) model

Action (O(N) model):

tuning parameter ("mass")

$$S = \int d^d\vec{x} \left[\frac{1}{2} \left(\vec{\nabla} \phi^a(\vec{x}) \right)^2 + \frac{r}{2} \left(\phi^a(x) \right)^2 + \frac{u}{4!} \left(\phi^a(x)^2 \right)^2 \right] \quad , a = 1, \dots, N$$

self-interaction coupling

$$= \int_0^\Lambda \frac{d^d\vec{k}}{(2\pi)^d} \, \frac{1}{2} \, \phi^a(-\vec{k}) \left(\vec{k}^2 + r \right) \phi^a(\vec{k}) + \frac{u}{4!} \int_0^\Lambda \frac{d^d\vec{k}_1 d^d\vec{k}_2 d^d\vec{k}_3}{(2\pi)^{3d}} \, \phi^a(\vec{k}_1) \phi^a(\vec{k}_2) \phi^b(\vec{k}_3) \phi^b(-\vec{k}_1 - \vec{k}_2 - \vec{k}_3)$$

where we have rescaled $\xi_0^2 (\phi^a)^2 \mapsto \phi^a$ and introduced an ultraviolet momentum cutoff Λ, $0 \leq |\vec{k}| \leq \Lambda$, with, e.g., $\Lambda \sim \frac{\pi}{a}$ (a: lattice constant).

Three stages of RG transformation:

1. Eliminate "fast" modes $\phi_>$ with momenta $\frac{\Lambda}{b} \leq |\vec{k}| < \Lambda$ ("momentum shell").

$$\phi(\vec{k}) \equiv \underset{\substack{\text{slow modes} \\ 0 \leq |\vec{k}| < \frac{\Lambda}{b}}}{\underset{\uparrow}{\Theta\left(\tfrac{\Lambda}{b} - |\vec{k}|\right) \phi_<(\vec{k})}} + \underset{\substack{\text{fast modes} \\ \frac{\Lambda}{b} \leq |\vec{k}| < \Lambda}}{\underset{\uparrow}{\Theta\left(|\vec{k}| - \tfrac{\Lambda}{b}\right) \phi_>(\vec{k})}}$$

2. Rescale momenta $\vec{k} \mapsto \vec{k}' = b\vec{k}$ with $0 \leq |\vec{k}'| < \Lambda$ for slow modes.

3. Introduce "renormalized" fields $\phi'(\vec{k}') = b^y \, \phi_<(\vec{k}'/b)$ with y chosen such that the new action in terms of ϕ' has the same coefficient for a certain quadratic term.

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RG for Gaussian model ($u=0$):

1. Mode elimination:

$$Z = \int \mathcal{D}\phi_< \int \mathcal{D}\phi_> \, e^{-S_0[\phi_<, \phi_>]}$$

with

$$S_0[\phi_<, \phi_>] = \int_0^{\Lambda/b} \frac{d^d\vec{k}}{(2\pi)^d} \, \frac{1}{2} \phi_<(-\vec{k}) (\vec{k}^2 + r) \phi_<(\vec{k}) + \int_{\Lambda/b}^{\Lambda} \frac{d^d\vec{k}}{(2\pi)^d} \, \frac{1}{2} \phi_>(-\vec{k}) (\vec{k}^2 + r) \phi_>(\vec{k})$$

Thus:

$$Z = \int \mathcal{D}\phi_< \, e^{-\underbrace{\int_0^{\Lambda/b} \phi_< (k^2 + r) \phi_<}_{\equiv S_{\text{eff}}}} \cdot \underset{\substack{\uparrow \\ \text{independent of } \phi_<}}{\text{const.}}$$

$$\left[Z_{0>} = \int \mathcal{D}\phi_> \, e^{-S_{0>}} = \left[\det_{\frac{\Lambda}{b} < k < \Lambda} (k^2 + r)\right]^{-1/2} \right]$$

2. Momentum rescaling:

$$S_{\text{eff}} = \int_0^{\Lambda/b} \frac{d^d\vec{k}}{(2\pi)^d}\ \frac{1}{2}\phi_<(-\vec{k})\,(k^2+r)\,\phi_<(\vec{k}) \left[\begin{array}{l} \vec{k}' = b\vec{k} \\ d^d\vec{k}' = b^d\, d\vec{k} \end{array}\right.$$

$$= \int_0^{\Lambda} \frac{d^d\vec{k}'}{(2\pi)^d}\ b^{-d}\ \frac{1}{2}\phi_<(-\vec{k}'/b)\,(b^{-2}k'^2+r)\,\phi_<(\vec{k}'/b)$$

3. Field renormalization:

with $\phi'(\vec{k}') = b^y\,\phi_<(\vec{k}'/b)$:

$$S_{\text{eff}} = \int_0^{\Lambda} \frac{d^d\vec{k}'}{(2\pi)^d}\ \frac{1}{2}\phi'(-\vec{k}')\left(b^{-d-2-2y}\,k'^2 + b^{-d-2y}\,r\right)\phi'(\vec{k}')$$

has the same form as original S for $y = -\frac{d+2}{2}$.

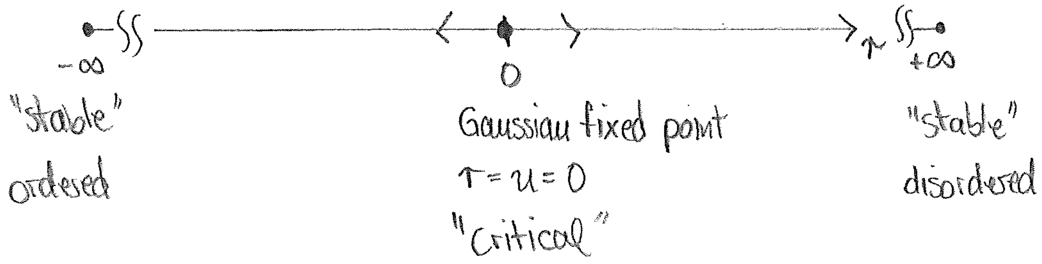
Then:

$$Z = \int D\phi'\ e^{-\int_0^{\Lambda} \frac{1}{2}\phi'(k'^2+r')\phi'} \qquad \text{with} \quad \underline{\underline{r' = b^2 r}}$$

Beta function ($0 < \ln b \ll 1$):

$$\beta_r = \frac{dr}{d\ln b} = 2r \qquad (\text{for } u=0)$$

RG flow diagram:



RG for ϕ^4 model ($u>0$ with $N=1$):

1. Mode elimination:

$$Z = \int \mathcal{D}\phi_< \int \mathcal{D}\phi_> \, e^{-S_{0<} - S_{0>} - S_{int}[\phi_<,\phi_>]}$$

$$= \int \mathcal{D}\phi_< \, e^{-S_{0<}} \int \mathcal{D}\phi_> \, e^{-S_{0>}} \left(1 - S_{int}[\phi_<,\phi_>] + \mathcal{O}(u^2)\right)$$

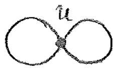
with

$$S_{int}[\phi_<,\phi_>] = \frac{u}{4!}\Bigg[\int_{0\,\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\Lambda/b} \phi_<\phi_<\phi_<\phi_< + \int_{\Lambda/b\,\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\Lambda} \phi_>\phi_>\phi_>\phi_>$$

$$+ \binom{4}{2} \int_{0\,\vec{k}_1,\vec{k}_2,\vec{k}_3}^{\Lambda} \phi_<\phi_<\phi_>\phi_> \Bigg] \qquad \left[\int_{0\,\vec{k}}^{\Lambda} \equiv \int_0^{\Lambda} \frac{d^d\vec{k}}{(2\pi)^d} \right]$$

Thus:

"tree-level"

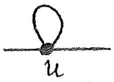


"vacuum"

$$Z = Z_{0>} \int \mathcal{D}\phi_< \, e^{-S_{0<}} \Bigg(1 - \frac{u}{4!} \Bigg[\int_0^{\Lambda/b} \underset{\uparrow}{\langle \phi_<\phi_<\phi_<\phi_< \rangle_{0>}} + \int_{\Lambda/b}^{\Lambda} \langle \phi_>\phi_>\phi_>\phi_> \rangle_{0>}$$

average w.r.t. $S_{0>}$

$\langle \ldots \rangle_{0>} \equiv \int \mathcal{D}\phi_> \, e^{-S_{0>}} (\ldots) / Z_{0>}$



$$+ \binom{4}{2} \int_0^{\Lambda} \langle \phi_<\phi_<\phi_>\phi_> \rangle_{0>} \Bigg] + \mathcal{O}(u^2) \Bigg)$$

Wick's theorem:

$$\langle \phi_<\phi_<\phi_>\phi_> \rangle_{0>} = \langle \phi_<\phi_< \rangle_{0>} \langle \phi_>\phi_> \rangle_{0>}$$

[exercise sheet 3, problem 1c]

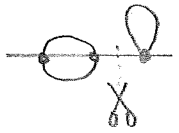
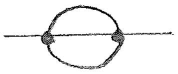
Feynman rules (momentum-shell RG):

- vertex $\overset{u}{\times} \quad \hat{=} \quad \frac{u}{4!}\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\,(2\pi)^d$
- internal line  $\hat{=} \quad \langle \phi_> \phi_> \rangle_{0>}$
- external line  $\hat{=} \quad \phi_<$

Remark:

Only one-particle irreducible (1PI) connected diagrams (that remain connected after cutting one internal line) contribute to the RG flow. [linked-clusters theorem]

Example:

 is not 1PI, but  is 1PI

Reexponentiation:

$$Z = Z_> \int D\phi_<\, e^{\underbrace{-S_{0<} - \frac{u}{4!}\left[\int_0^{\Lambda/b} \phi_<\phi_<\phi_<\phi_< + \binom{4}{2}\left(\int_0^{\Lambda/b} \phi_<\phi_<\right)\left(\int_{\Lambda/b}^{\Lambda} \frac{1}{k^2+r}\right)\right] + \mathcal{O}(u^2)}_{=:-S_{\text{eff}}}}$$

with $\int_{\Lambda/b}^{\Lambda} \frac{1}{k^2+r} = \frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}\ln b + \mathcal{O}(\ln^2 b)$

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2. Momentum rescaling: $\vec{k}' = b\vec{k}$

$$S_{\text{eff}} = S_{0<} + \frac{u}{4!}\left[\int_0^{\Lambda} b^{-3d}\,\phi_<\phi_<\phi_<\phi_< + \binom{4}{2}\int_0^{\Lambda} b^{-d}\,\phi_<\phi_<\,\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}\ln b\right] + \mathcal{O}(u^2, \ln^2 b)$$

3. Field renormalization: $\phi'(\vec{k}') = b^{-\frac{d+2}{2}} \phi_<(\vec{k}'/b)$

$$S_{eff} = \int_0^{\Lambda} \frac{1}{2} \phi'(\vec{k}'^2 + \underbrace{b^2 r + \frac{u}{2} \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{\Lambda^2+r} \ln b}_{\equiv r'})\phi'$$

$$+ \int_0^{\Lambda} \underbrace{\frac{u}{4!} b^{4-d}}_{\frac{u'}{4!}} \phi'\phi'\phi'\phi' \quad + O(u^2, \ln^2 b) \qquad [b^2 = 1 + O(\ln b)]$$

Then:

$$r' = b^2 r + \frac{u}{2} \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{\Lambda^2+r} \ln b + O(u^2, \ln^2 b)$$

$$u' = b^{4-d} u + O(u^2, \ln^2 b)$$

Introduce dimensionless variables (for convenience):

$$r \mapsto t \equiv \frac{r}{\Lambda^2}$$

$$u \mapsto g \equiv \frac{S_d}{(2\pi)^d} \frac{u}{\Lambda^{4-d}}$$

Beta functions:

$$\boxed{\begin{aligned} \beta_t &= \frac{dt}{d\ln b} = 2t + \frac{g}{2}\frac{1}{1+t} + O(g^2) \\ \beta_g &= \frac{dg}{d\ln b} = (4-d)g + O(g^2) \end{aligned}}$$

Remarks:

- Scaling dimensions $\dim[r] = 2$ and $\dim[u] = 4-d$ agree with power-counting dimensions:

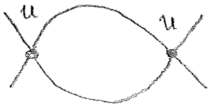
$$0 = [S] = \underbrace{[\nabla^2]}_{2} + [\phi^2] + \underbrace{[d^dx]}_{-d} \quad \Rightarrow \quad [\phi] = \frac{d-2}{2}, \quad [r] = 2 = \dim[r] \ ✓$$

inverse-length dimension

$$0 = [S] = [u] + \underbrace{[\phi^4]}_{2(d-2)} + \underbrace{[d^dx]}_{-d} \quad \Rightarrow \quad [u] = 4-d = \dim[g] \ ✓$$

- To compute the leading interaction correction to β_g we need to compute the g^2 contribution.

Leading interaction correction to β_g (diagrammatically):



$$= \underset{\substack{\uparrow \\ \text{re-exponentiation}}}{(-1)} \; \underset{\substack{\nwarrow \\ \text{expansion of } \exp(..)}}{\frac{1}{2!}} \left(\frac{-u}{4!}\right)^2 (\phi\phi\underbrace{\phi\underbrace{\phi)(\phi}\phi}\phi\phi) \times \binom{4}{2}^2 \cdot 2$$

where $\underbrace{\phi\phi} \equiv \langle \phi_> \phi_> \rangle_{0>}$

$$= -\frac{1}{4!}\,\frac{3}{2}u^2\,\phi\phi\phi\phi \int_{\Lambda/b}^{\Lambda} \frac{1}{(k^2+r)^2}$$

Thus:

$$\boxed{\beta_g = (4-d)g - \frac{3}{2}g^2\frac{1}{(1+t)^2} + \mathcal{O}(g^3)}$$

Generalization to O(N) model:

$$\boxed{\begin{aligned} \beta_t &= \frac{dt}{d\ln b} = 2t + \frac{N+2}{6}\,\frac{g}{1+t} + \mathcal{O}(g^2) \\ \beta_g &= \frac{dg}{d\ln b} = (4-d)g - \frac{N+8}{6}\,\frac{g^2}{(1+t)^2} + \mathcal{O}(g^3) \end{aligned}}$$

Fixed points

(a) Gaussian fixed point: $t^* = g^* = 0$

Near this fixed point:

$\dim[u] = 4-d$ irrelevant for $d>4$

$\dim[u_6] = 6-2d$ irrelevant for $d>3$

$\uparrow$

ϕ^6 coupling

(b) Wilson - Fisher fixed point: Assume $\varepsilon = 4-d \ll 1$ "fractional dimension"

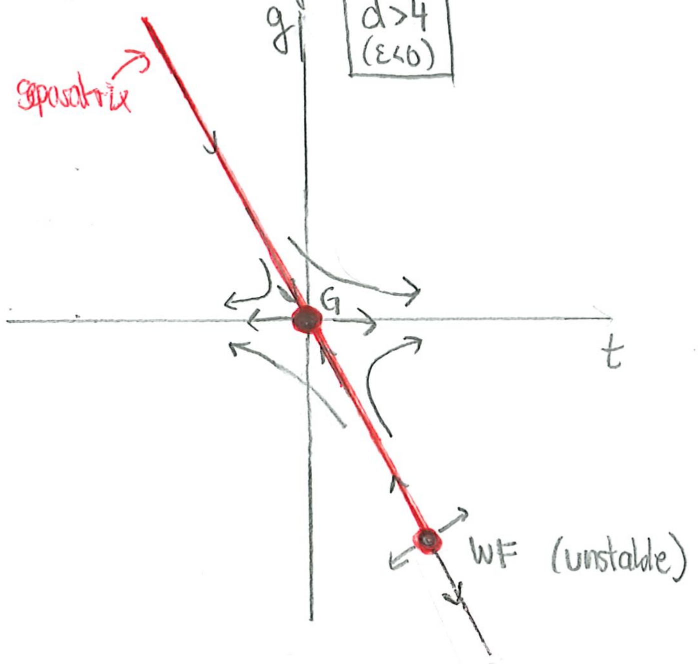
$$g^* = \frac{6}{N+8}\varepsilon + \mathcal{O}(\varepsilon^2)$$

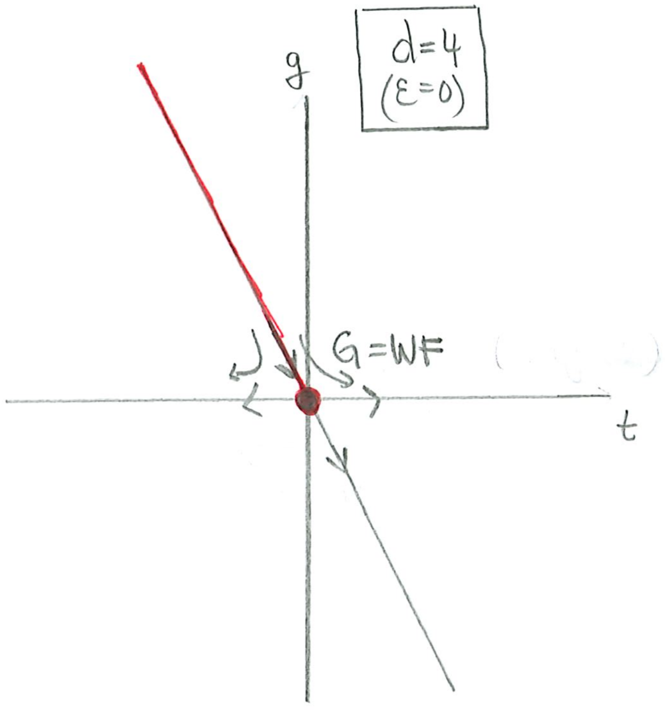
$$t^* = -\frac{N+2}{2(N+8)}\varepsilon + \mathcal{O}(\varepsilon^2)$$

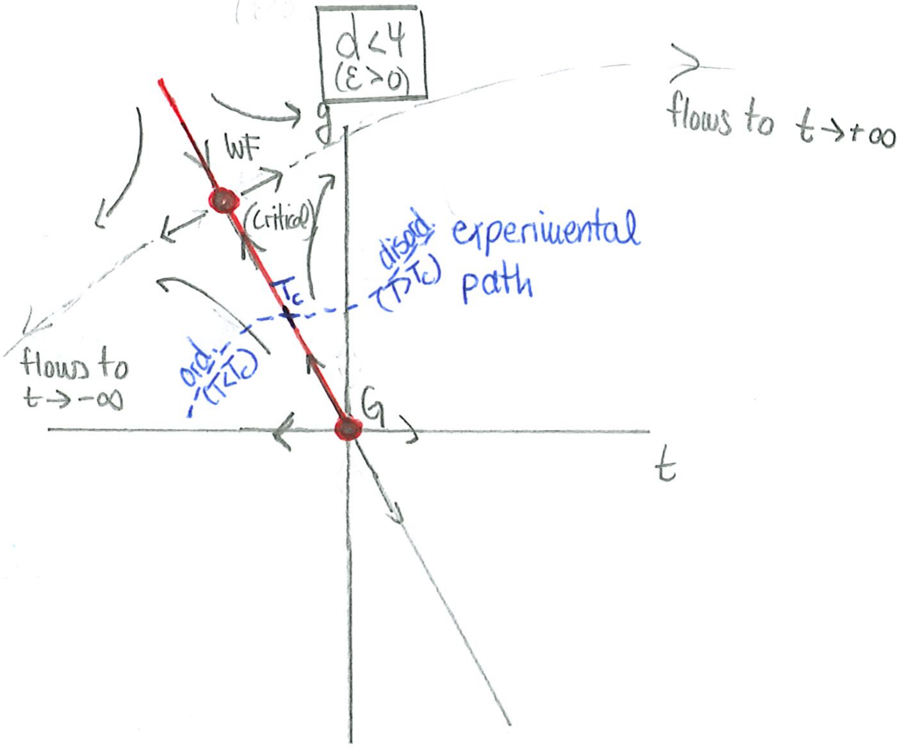
Remark:

Systematic loop expansion: contributions at $\mathcal{O}(\varepsilon^n)$ arise from n-loop Feynman diagrams

RG flow diagrams :







Remarks:

- The Gaussian (Wilson-Fisher) fixed governs the critical behavior for $d>4$ ($d<4$).
- $d = d_c^+ = 4$ is the upper critical dimension.

- For $d > d_c^+$ Landau theory becomes (asymptotically) exact because the theory is effectively Gaussian at criticality.
- An experimental system at T_c flows to the respective critical fixed point and the system becomes scale invariant.
- The critical behavior is governed by the flow in the vicinity of the critical fixed point.

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Perturbations to the Wilson-Fisher fixed point

Consider RG flow near the WF fixed point:

$$t = t^* + \delta t$$
$$g = g^* + \delta g$$

with $\delta t \ll t^*$ and $\delta g \ll g^*$.

Linearized flow equations (O(N) model):

$$\frac{d}{d\ln b}\begin{pmatrix}\delta t\\ \delta g\end{pmatrix} = \underbrace{\begin{pmatrix} 2-\varepsilon\frac{N+2}{N+8} & \frac{N+2}{6}\left(1+\varepsilon\frac{N+2}{2(N+8)}\right)\\ 0 & -\varepsilon\end{pmatrix}}_{=:\,(B_{ij})}\begin{pmatrix}\delta t\\ \delta g\end{pmatrix} + \mathcal{O}(\delta^2)$$

"stability matrix"

Diagonalization of stability matrix:

$$\sum_{j=1}^{2} B_{ij}\,\underset{\substack{\uparrow\\ \text{eigenvectors}}}{v_j^I} = \underset{\substack{\uparrow\\ \text{eigenvalues}}}{\theta^I}\, v_i^I \qquad I = 1,2 \quad (\text{no sum!})$$

Remarks:

- $\theta^I = \dim[v^I]$ is the <u>scaling dimension</u> of the coupling v^I <u>at the WF fixed point</u>
- Any <u>critical</u> fixed point has exactly one $\theta^I > 0$ (w.l.o.g. for $I=1$)

Integration of the relevant direction:

$$v^1(b) = v^1(0)\, b^{\theta^1} \qquad \text{with } \theta^1 > 0$$

Scaling transformation of reduced temperature t_{red} (& other tuning parameters):

$$t_{red} \sim \delta t \sim v^1 \quad \Rightarrow \quad t_{red} \mapsto b^{\theta_1}\, t_{red} \quad \Rightarrow \quad \theta_1 \equiv y_t$$

Correlation-length exponent:

$$\boxed{\nu = \frac{1}{\theta^1}}$$

Wilson-Fisher fixed point:

$$\nu = \frac{1}{2} + \frac{N+2}{4(N+8)}\varepsilon + \mathcal{O}(\varepsilon^2)$$

Gaussian fixed point:

$$\nu = \frac{1}{2}$$

Remarks:

- For $N=1$ and $\varepsilon=1$ we get

$$\nu = \frac{1}{2} + \frac{1}{12} \pm \dots \approx 0.58$$

- More advanced techniques (3D Ising universality):

$\nu = 0.629(3)$ (six-loop ε expansion + Borel summation)

$\nu = 0.631(4)$ (high-temperature expansion)

$\nu = 0.6300(1)$ (MC simulation)

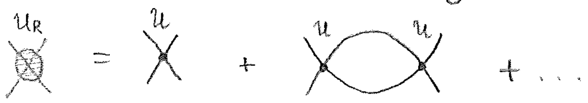
$\nu = 0.64(1)$ (neutron scattering of FeF_2)
[Ising AFM]

[Guida & Zinn-Justin,
J. Phys. A 31, 8103 (1998)]

4.4 Field-theoretical perspective and anomalous dimension

Idea: Perturbation theory in "renormalized" coupling u_R:

$$u_R = u - \frac{N+8}{6} u^2 \int_0^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+r)^2} + \mathcal{O}(u^3)$$



Remarks:

- u_R is the effective coupling after integrating out all modes.
- Dimensionless coupling:

$$u \mapsto g \equiv \frac{S_d}{(2\pi)^d} \frac{u}{|r/t|^{(4-d)/2}}$$ diverges for $r \to 0$ when $d < d_c^+ = 4$

$\Rightarrow$ standard perturbation theory (in u) breaks down at criticality

- "Renormalized" perturbation theory (in u_R) can be set up to yield finite result.

Example (anomalous dimension, sketch):

Expected critical correlator:

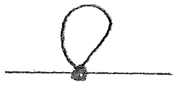
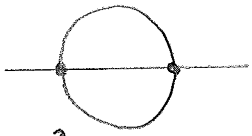
$$\langle \phi(-k)\phi(k)\rangle \propto \frac{1}{k^{2-\eta}}$$

anomalous dimension

Standard perturbation theory:

$$\langle \phi(-k)\phi(k)\rangle \propto \frac{1}{k^2+\tau-\Sigma(k)}$$

with the "self-energy"

$\Sigma(k) =$  $+$  $+ \dots$

"tadpole"
contributes only
to $\Sigma(0)$

"sunset"
leading nontrivial
momentum dependence

Critical point: $\tau_R = \tau - \Sigma(0) = 0$

Sunset diagram yields in $D = 4-\varepsilon$:

$$\Sigma(k) - \Sigma(0) = u^2\left[\underset{\uparrow\ \text{constant}}{c_1} k^2 \ln\left(\frac{\Lambda}{k}\right) + O(k^4,\varepsilon)\right] + \dots$$

To the leading order $u_R = u + O(u^2)$ and thus

$$\langle \phi(-k)\phi(k)\rangle \propto \frac{1}{k^2\left[1 + c_2\, g_R^2 \ln\left(\frac{\Lambda}{k}\right)\right]} + O(g_R^3)$$

$$= \frac{1}{k^2}\left(\frac{\Lambda}{k}\right)^{-c_2 g_R^2} + O(g_R^3) \qquad \left[k^x = 1 + x\ln k + O(x^2)\right]$$

with $g_R = g^*$ at the critical point.

Reinstating the constants, we read off

$$\eta = c_2 (g^*)^2 = \frac{N+2}{2(N+8)^2}\,\varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

Remarks:

- The last step $\quad 1 + c_2 g_R^2 \ln\left(\frac{\Lambda}{k}\right) = \left(\frac{\Lambda}{k}\right)^{c_2 g_R^2} + \mathcal{O}(g_R^4)$ effectively resums an infinite number of diagrams
- For $N=1$ and $\varepsilon = 1$ (3D Ising):

$$\eta = \frac{1}{54} + \ldots \approx 0.02$$

to be compared with (almost exact) value from MC

$$\eta_{MC} = 0.0363(1)$$

5.6.26

4.5 Phase transitions and critical dimensions

Universality: different microscopic models flow to the same RG fixed point at criticality

Critical dimensions:

- Upper critical dimension d_c^+ : Mean-field theory asymptotically exact for $d \geq d_c^+$
- Lower critical dimension d_c^- : Fluctuations destroy ordered phase at any temperature for $d \leq d_c^-$
- Critical exponents typically depend on d for $d_c^- < d < d_c^+$ and become d-independent for $d > d_c^+$ [exception: system with sufficiently long-ranged interactions].

- Classical magnets with short-range interactions [O(N) models]:

$$d_c^+ = 4 \quad \text{and} \quad d_c^- = \begin{cases} 2 & \text{for } N > 2 \\ 1 & \text{for } N = 1 \end{cases}$$

(The case $N=2$ and $d=2$ is special.)

Physics near upper critical dimension [O(N) models]:

- For $d < d_c^+ = 4$: critical fixed point = Wilson-Fisher fixed point, observables computable in renormalized perturbation theory in $u^* = u^*(d)$, hyperscaling valid.

- For $d > d_c^+ = 4$: critical fixed point = Gaussian fixed point, observables computable in standard perturbation theory in u, exponents take mean-field values, e.g. $\alpha = 0$, $\eta = 0$, $\nu = \frac{1}{2}$, etc. hyperscaling violated: e.g.

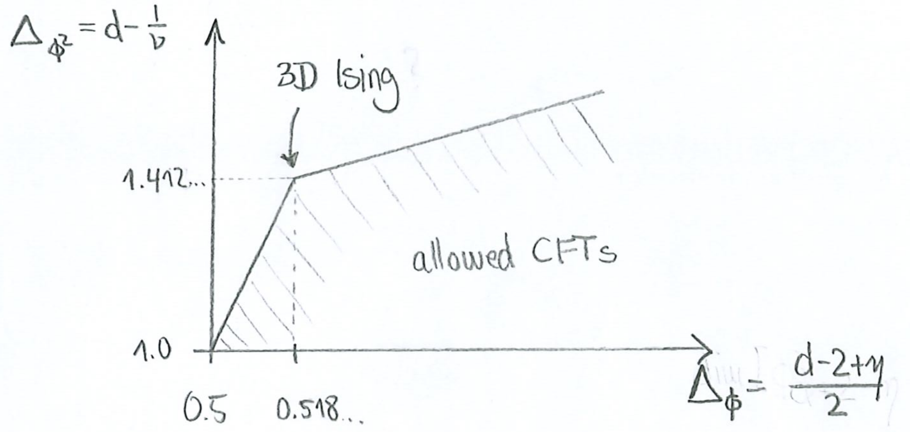
$$2 - \alpha \neq d\nu \qquad \text{(Josephson)}$$

can be traced back to presence of dangerously irrelevant coupling u (free energy nonanalytic at $u = u^* = 0$.)

- For $d = d_c^+ = 4$: logarithmic corrections to mean-field behavior

Analytical alternatives to $\varepsilon = 4-D$ expansion:

- $\frac{1}{N}$ expansion (exercise sheet 3)
- $2+\varepsilon$ expansion : expansion in $T_c(\varepsilon) \propto O(\varepsilon)$
- Conformal bootstrap : use symmetry and unitarity arguments to constrain scaling dimensions of operators assuming conformal invariance



world record in precision, e.g.:

$\nu = 0.629971(4)$ (3D Ising, Kos et al., 2016)
[tinyurl.com/ising-bs]

Summary: