

Exercises for “Quantum Phase Transitions”

Summer 26

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Exercise 2 (11.05.26)

1. Tricritical point in an antiferromagnet

(6 points)

In an antiferromagnet, the external magnetic field h couples to the total magnetization m instead of the antiferromagnetic order parameter, the staggered magnetization n . Assume that the coupling between m and n is described phenomenologically by the Landau free energy density

$$f(n, m) = \frac{t}{2}n^2 + \frac{b}{4}n^4 + \frac{v}{2}m^2 + \frac{w}{2}n^2m^2 - hm, \quad (1)$$

where $t = (T - T_0)a$, and T_0, a, b, v, w are positive constants.

- (a) Show that this model features a paramagnetic phase with magnetization $m_0 > 0$ for $h > 0$. Derive a temperature-independent relation $m = m(n^2)$ between the magnetization m and the staggered magnetization n in the antiferromagnetic phase, i.e., for $n^2 > 0$.

Hint: Equilibrium states are given by minima of $f(n, m)$.

- (b) Consider the antiferromagnetic phase near the phase transition, assuming small values of n^2 . Write $m = m_0 + \delta m$, expand $m(n^2)$ for small n^2 , and derive a relation between δm and n^2 .
- (c) Show that the effective free energy density for the staggered magnetization $g(n) = f(n, m_0 + \delta m) - f(0, m_0)$ can be written as

$$g(n) = \frac{\bar{a}}{2}n^2 + \frac{\bar{b}}{4}n^4 + \frac{\bar{c}}{6}n^6 + \mathcal{O}(n^8), \quad (2)$$

with to-be-determined temperature- and field-dependent coefficients \bar{a} , \bar{b} , and \bar{c} .

- (d) Show that the model (1) features a tricritical point at temperature T_t and field h_t , where

$$T_t = T_0 - \frac{bv}{2aw}, \quad h_t^2 = \frac{bv^3}{2w^2}. \quad (3)$$

Hint: Here we define a tricritical point as a point where first- and second-order transition lines meet. Depending on the sign of the coefficient \bar{b} in $g(n)$, the transition between the paramagnetic and antiferromagnetic phases are first or second order, cf. Problem 1 on Exercise 1.

- (e) Show that the second-order phase transition occurs for $h < h_t$ at

$$T_c = T_0 - \frac{wh^2}{av^2}, \quad (4)$$

and the first-order transition occurs for $h > h_t$ at

$$T_c = T_0 - \frac{3wh^2}{4av^2} - \frac{bv}{4aw} + \frac{b^2v^4}{16aw^3h^2}. \quad (5)$$

- (f) Sketch the phase diagram in the (T, H) plane.

please turn over!

2. Scaling hypothesis

(4 points)

Consider the static scaling hypothesis for the free energy density

$$f_s(t, h) = b^{-d} f_s(b^{y_t} t, b^{y_h} h) \quad (6)$$

with scaling exponents y_t for the reduced temperature t and y_h for the external field h .

(a) Use the above scaling hypothesis to derive the relation

$$\delta = \frac{d + 2 - \eta}{d - 2 + \eta} \quad (7)$$

between the critical-isotherm exponent δ and the anomalous dimension η .

Hint: Use the relation $y_t = 1/\nu$ and Fisher's law $\gamma = \nu(2 - \eta)$ derived in class.

(b) In principle, critical exponents above and below a transition could differ from each other. Show for the example of the correlation-length exponents ν and ν' above and below the transition, respectively, that the static scaling hypothesis for the free energy density implies that they are equal,

$$\nu(T > T_c) = \nu'(T < T_c). \quad (8)$$

Hint: For fixed $h \neq 0$, $f_s(t, h)$ should be a smooth function of t , because the only singularity that we expect is at $t = h = 0$. Show that $f_s(t, h)$ can be written in the form

$$f_s(t, h) = h^{d/y_h} F_f^\pm \left(\frac{|t|}{h^{1/(\bar{\nu} y_h)}} \right), \quad \text{with } \bar{\nu} = \begin{cases} \nu & \text{for } T > T_c \\ \nu' & \text{for } T < T_c \end{cases}, \quad (9)$$

and explain how the smoothness assumption mentioned above constrains the analytic form of the function $F_f^\pm(x)$.