Exercises for "Quantum Phase Transitions" **Summer 26**

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1. Generating functional for noninteracting real bosons (4 points)

Consider the (discretized) field theory of a noninteracting real scalar boson field $\varphi \equiv (\varphi_k)_{k=1}^M$ with action

$$S[\varphi] = \sum_{k,l=1}^{M} \frac{1}{2} \varphi_k K_{kl} \varphi_l, \tag{1}$$

and positive definite symmetric and real matrix $K = K^\top$ ("kernel").

(a) Show that the n-point correlation functions

$$\langle \varphi_{l_1} \varphi_{l_2} \dots \varphi_{l_n} \rangle := \frac{1}{Z[0]} \int \prod_{k=1}^{M} \frac{d\varphi_k}{\sqrt{2\pi}} \varphi_{l_1} \varphi_{l_2} \dots \varphi_{l_n} \exp(-S[\varphi]) \tag{2}$$

can be obtained from the *generating functional*

$$Z[h] = \int \prod_{k=1}^{M} \frac{d\varphi_k}{\sqrt{2\pi}} \exp\left(-S[\varphi] + \sum_{k=1}^{M} h_k \varphi_k \right) \tag{3}$$

via suitable derivatives with respect to the external source ("magnetic field") h.

(b) Show that the generating functional for a noninteracting real scalar boson field theory can be computed in closed form as

$$Z[h] = (\det K)^{-1/2} \exp\left(\sum_{k,l=1}^{M} \frac{1}{2} h_k (K^{-1})_{kl} h_l \right). \tag{4}$$

(c) Use the above result to show that the propagator $G^{(2)}_{kl} \equiv \langle \varphi_k \varphi_l \rangle$ and the four-point function $G^{(4)}_{klmn} = \langle \varphi_k \varphi_l \varphi_m \varphi_n \rangle$ can be written as

$$G^{(2)}_{kl} = (K^{-1})_{kl} \qquad \text{and} \qquad G^{(4)}_{klmn} = G^{(2)}_{kl} G^{(2)}_{mn} + G^{(2)}_{km} G^{(2)}_{ln} + G^{(2)}_{kn} G^{(2)}_{lm}. \tag{5}$$

2. Partition function for complex bosons (1 point)

Use the result of Problem 1 to show that the partition function $Z \equiv Z[0]$ for the theory of noninteracting complex boson fields Φ, Φ^* is

$$Z = \int \prod_{k=1}^{M} \frac{d\phi_k^* d\phi_k}{2\pi \mathrm{i}} \exp\left(-\sum_{k,l=1}^{M} \phi_k^* K_{kl} \phi_l \right) = (\det K)^{-1}, \tag{6}$$

assuming a positive definite Hermitian kernel $K = K^\dagger$.

please turn over!

3. Susceptibility exponent γ in the large-N limit (5+3* points)

Consider the partition function for the theory of N complex boson fields Φ_a and Φ_a^*, $a = 1, \ldots, N$, interacting via an ultralocal two-body interaction,

$$Z = \int \prod_{a=1}^{N} \mathcal{D}\Phi_a^*(\vec{x})\mathcal{D}\Phi_a(\vec{x})\, \mathrm{e}^{-S[\Phi^*,\Phi]} \tag{7}$$

with action

$$S[\Phi^*,\Phi] = \int d^d\vec{x} \left[\sum_{a=1}^{N} \left(|\nabla\Phi_a(\vec{x})|^2 + t|\Phi_a(\vec{x})|^2 \right) + \frac{\lambda}{2N} \left(\sum_{a=1}^{N} |\Phi_a(\vec{x})|^2 \right)^2 \right]. \tag{8}$$

t is the tuning parameter for a classical phase transition distinguishing the disordered phase for $t > t_\mathrm{c}$ from an ordered phase for $t < t_\mathrm{c}$. λ denotes the quartic coupling.

(a) Show that the partition function can be written as

$$Z = \int \prod_{a=1}^{N} \mathcal{D}\Phi_a^*(\vec{x})\mathcal{D}\Phi_a(\vec{x})\mathcal{D}\sigma(\vec{x})\, \mathrm{e}^{-S_0[\Phi^*,\Phi] - \int d^d\vec{x} \left[\frac{N}{2\lambda}\sigma^2(\vec{x}) + \mathrm{i}\sigma(\vec{x})|\Phi(\vec{x})|^2\right]}, \tag{9}$$

where we have introduced the composite field $\sigma(\vec{x})$ which couples to $|\Phi(\vec{x})|^2 \equiv \sum_{a=1}^{N} |\Phi_a(\vec{x})|^2$ and S_0 denotes the Gaussian part of the action S. (This is the so-called *Hubbard-Stratonovich* transformation.)

(b) Integrate over all components Φ_a, $2 \le a \le N$, except the first one, to obtain an effective theory in Φ_1 and σ. Consider the limit $N \to \infty$, argue that the saddle-point approximation discussed in class becomes exact in this limit, and use it to compute the free energy density.

Hint: The resulting free energy density reads

$$\frac{f}{Nk_\mathrm{B}T} = (t+\sigma)|\Phi_1|^2 - \frac{\sigma^2}{2\lambda} + \frac{1}{V}\int \frac{d^d\vec{k}}{(2\pi)^d} \ln(k^2 + t + \sigma),$$

with the saddle-point conditions

$$(t+\sigma)\Phi_1 = 0 \qquad \text{and} \qquad \sigma = \lambda \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{k^2 + t + \sigma} + \lambda|\Phi_1|^2,$$

where $V := \int \mathrm{d}^d\vec{x}$ is the spatial volume, and we have assumed uniform fields $|\Phi_1|^2 := \frac{1}{V}\int \mathrm{d}^d\vec{x}\, |\Phi_1(\vec{x})|^2$ and $\sigma := \frac{1}{V}\int \mathrm{d}^d\vec{x}\, \sigma(\vec{x})$ at the saddle point, rotated $\mathrm{i}\sigma \mapsto \sigma$, and rescaled $\Phi_1/\sqrt{N} \mapsto \Phi_1$.

(c) Show that the theory exhibits a phase transition for $d > 2$ at $t_\mathrm{c} = -\lambda \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{k^2}$ and that the inverse susceptibility $\chi^{-1} \propto t + \sigma$ satisfies in the disordered phase $\Phi_1 = 0$ the implicit equation

$$(t+\sigma)\left(1 + \lambda \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{k^2(k^2+t+\sigma)}\right) = t - t_\mathrm{c}, \tag{10}$$

for $t > t_\mathrm{c}$. What happens for $d \le 2$?

(d) Assume an ultraviolet cutoff Λ in the integral over wavevectors and compute the scaling form of the susceptibility in the critical region $t + \sigma \to 0$ for (i) $d > 4$, (ii) $d = 4$, and (iii) $2 < d < 4$. Compare with the predictions from Landau theory for the original model in Eq. (7).

Hint: (i) $\chi \propto |t - t_\mathrm{c}|^{-1}$, (ii) $\chi \propto \frac{\ln|t-t_\mathrm{c}|}{|t-t_\mathrm{c}|}$, (iii) $\chi \propto |t - t_\mathrm{c}|^{-2/(d-2)}$.