Exercises for "Quantum Phase Transitions" **Summer 26**

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1. Interchange of limits in the classical Ising chain (7 points)

Consider a classical Ising chain with M sites and ferromagnetic nearest-neighbor exchange $K = J/(k_\mathrm{B}T) > 0$:

$$H = -\sum_i K\sigma_i\sigma_{i+1}. \tag{1}$$

(a) Show that the partition function

$$Z \equiv \sum_{\{\sigma_i=\pm1\}} e^{-H} \tag{2}$$

can be written as $Z = \operatorname{Tr} T^M$ with the transfer matrix

$$T = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}. \tag{3}$$

(b) Evaluate Z as well as the spin-spin correlation function $\langle\sigma_i\sigma_0\rangle$ exactly.

(c) Investigate now two possible routes to obtain the correlation function in the limit of large K (small T).

(i) Approximate $\langle\sigma_i\sigma_0\rangle$ first for large K and then take the limit $M \to \infty$.

(ii) Take first the thermodynamic limit $M \to \infty$ and then approximate $\langle\sigma_i\sigma_0\rangle$ for large K.

(d) Why are the results different? Explain which different physical situations the two routes correspond to.

Hint: Determine the energy Δ required to create a domain wall between a region with all spins up and a region with all spins down, and think in terms of domain walls.

(e) Show that the correlation length in the thermodynamic limit ($M \to \infty$) satisfies the relation $\xi = (a/2)e^{\Delta}$ for large K, where Δ is the energy required to create a domain wall between a region with all spins up and a region with all spins down.

2. Relation between energy gap and correlation length (3 points)

We wish to show now that the relationship

$$\xi = \frac{a}{2}e^{\Delta} \tag{4}$$

holds quite generally in a one-dimensional classical spin model, i.e., independently of the model and, to some extent, the temperature. To this end, we think of the spin configurations in terms of domain walls and assume the domain walls to be statistically uncorrelated from each other (i.e., we neglect possible interactions between the domain walls).

(a) Argue that the density of domain walls is given by $\rho = (1/a)e^{-\Delta}$. Consequently, it is sufficient to show $\xi = 1/(2\rho)$.

(b) Consider a long chain of length $Ma \gg \xi$ with $N = \rho Ma$ domain walls. The probability that any given domain wall is between 0 and $x > 0$ is $q = x/(Ma)$. Use the statistical independence of the domain walls to argue that

$$\langle \sigma(x)\sigma(0) \rangle = \sum_{j=0}^{N} (-1)^j q^j (1-q)^{N-j} \frac{N!}{j!(N-j)!}. \tag{5}$$

(c) Evaluate the above expression in the limit $N, M \to \infty$, while $\rho = N/(Ma)$ is finite, to show the desired result.