

## Exercises for “Quantum Phase Transitions”

Summer 26

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Exercise 6 (06.07.26)

**1. Quantum Ising chain with second-neighbor exchange**

(4 points)

Consider a quantum Ising chain with second-neighbor exchange in a transverse field,

$$H_I = - \sum_n (J\sigma_n^z\sigma_{n+1}^z + J_2\sigma_n^z\sigma_{n+2}^z + Jg\sigma_n^x). \quad (1)$$

Here, the spin-1/2 operators are represented by Pauli matrices  $\sigma_n^x$  and  $\sigma_n^y$  that fulfill the algebra:

$$\begin{aligned} \sigma_n^z\sigma_n^z &= \sigma_n^x\sigma_n^x = 1, \\ \sigma_n^z\sigma_n^x &= -\sigma_n^x\sigma_n^z, \\ \sigma_n^z\sigma_m^x &= \sigma_m^x\sigma_n^z, \quad \text{for } m \neq n. \end{aligned} \quad (2)$$

- Determine the dispersion relation of a domain-wall excitation to lowest order in  $g$ .
- Determine the dispersion relation of a flipped-spin excitation in the limit  $g \gg 1$ .
- Interpret the results.

**2. Self-duality of the quantum Ising chain**

(6 points)

We wish to derive the dual representation of the one-dimensional quantum Ising chain in a transverse field,

$$H_I = -J \sum_n (\sigma_n^z\sigma_{n+1}^z + g\sigma_n^x). \quad (3)$$

- First, introduce spin operators on the dual lattice, i.e., the lattice where the sites are given by the bonds of the original lattice,

$$\begin{aligned} \tau_n^x &= \sigma_{n+1}^z\sigma_n^z, \\ \tau_n^z &= \prod_{m \leq n} \sigma_m^x, \end{aligned} \quad (4)$$

and show that they satisfy the algebra (2) as well.

- Express the Hamiltonian (3) in terms of the dual operators.
- Use the dual Hamiltonian to derive a relation between the energy eigenvalues at coupling  $g$  and coupling  $1/g$ .
- Argue that the critical point of  $H_I$  is at  $g = 1$ . Which further assumptions are needed?