4. Exercise sheet for Theoretical Femtosecond Physics

PD Dr. Frank Großmann, Summer 2017

1*: *Classical minimal coupling*

Study classical minimal coupling by answering the following questions:

a) Under which conditions for the potentials $\mathbf{A}$ and Φ does the classical Lagrangian

$$L(\dot{\mathbf{r}}, \mathbf{r}, t) = \frac{m}{2}\dot{\mathbf{r}}^2 - q\Phi(\mathbf{r}, t) + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

lead to Newton's equations of motion with the Lorentz force?

b) Give explicit expressions for the canonical momentum $\mathbf{p} = \partial L/\partial\dot{\mathbf{r}}$ and for the mechanical momentum $\mathbf{p}_\mathrm{m} = m\dot{\mathbf{r}}$.

c) What is the explicit form of the Hamiltonian $H(\mathbf{p}, \mathbf{r}, t) = \dot{\mathbf{r}} \cdot \mathbf{p} - L(\dot{\mathbf{r}}, \mathbf{r}, t)$?

2: *Probability current density*

Find the modified expression for the probability current density $\mathbf{j}$ in the case of coupling of the motion of a charged particle to an external field. Show that the expression you gained is gauge invariant.

3: *Energy operator*

Let $\hat{\Theta}(\mathbf{A}, \Phi)$ be an operator that depends on the potentials of the electromagnetic field.

a) Show that for the operator $\hat{\Theta}$ to have a gauge invariant expectation value

$$\mathrm{e}^{\mathrm{i}\frac{q}{\hbar}\chi}\Theta(\mathbf{A}, \Phi)\mathrm{e}^{-\mathrm{i}\frac{q}{\hbar}\chi} = \hat{\Theta}(\mathbf{A}', \Phi')$$

has to hold.

b) Show that $\hat{H} = \frac{(\hat{\mathbf{p}} - q\mathbf{A})^2}{2m} + V(\mathbf{r}) + q\Phi$ is not a gauge invariant operator and its expectation value cannot be the energy. Discuss an alternative, that may be considered as the energy operator.

4*: *Kramers-Henneberger transformation*

Show that the two unitary transformations into the Kramers-Henneberger frame eleminate the terms proportional to $\mathbf{A}^2$ and $\mathbf{A}$ in the Hamiltonian. Due to the fact that the first transformation is a global phase transformation it just remains to calculate

$$\hat{U}_2\hat{V}\hat{U}_2^{-1}$$

to prove the shift in the argument of the potential.

Hint: Use the operator relation known as Baker-Haussdorff (or Hadamard) lemma $\mathrm{e}^{\hat{L}}\hat{M}\mathrm{e}^{-\hat{L}} = \sum_{n=0}^{\infty}\frac{1}{n!}[\hat{L}, \hat{M}]_n$, where $[\hat{L}, \hat{M}]_n = [\hat{L}, [\hat{L}, \hat{M}]_{n-1}]$ and $[\hat{L}, \hat{M}]_0 = \hat{M}$.

5*: *Magnetic field*

Investigate under which conditions the use of the electric field only in field-matter coupling is justified. Delineate regions in frequency intensity parameter space where a treatment of terms

a) proportional to v/c

b) proportional to $(v/c)^2$

becomes necessary

Hint: Use the reference H. R. Reiss, Phys. Rev. A **63**, 013409 (2000)

to be discussed on 31.05.2017

exercises with an asterisk may be handed in for examination purposes